Question 1

Worked Solution

Particle P moves with constant velocity $(-3\mathbf{i} + 2\mathbf{j})\ \mathrm{m\,s^{-1}}$.

At $t = 6$, position vector is $(-4\mathbf{i} - 7\mathbf{j})$ m.

Find distance from origin at $t = 2$.

Using $\mathbf{r}(t) = \mathbf{r}(6) + (t - 6)\mathbf{v}$ (constant velocity):

$$\mathbf{r}(2) = (-4\mathbf{i} - 7\mathbf{j}) + (2 - 6)(-3\mathbf{i} + 2\mathbf{j})$$

$$= (-4\mathbf{i} - 7\mathbf{j}) + (-4)(-3\mathbf{i} + 2\mathbf{j})$$

$$= (-4 + 12)\mathbf{i} + (-7 - 8)\mathbf{j} = 8\mathbf{i} - 15\mathbf{j}$$

$$|\mathbf{r}(2)| = \sqrt{64 + 225} = \sqrt{289} = 17 \text{ m}$$

Distance from origin at $t = 2$ is 17 m.

Question 2

Worked Solution

Constant acceleration $\mathbf{a} = (2\mathbf{i} - 5\mathbf{j})\ \mathrm{m\,s^{-2}}$. At $t = 0$, speed $= u$. At $t = 3$, $\mathbf{v} = (-6\mathbf{i} + \mathbf{j})\ \mathrm{m\,s^{-1}}$.

Find u.

Let initial velocity be $\mathbf{u}_0$. Using $\mathbf{v} = \mathbf{u}_0 + \mathbf{a}t$:

$$-6\mathbf{i} + \mathbf{j} = \mathbf{u}_0 + 3(2\mathbf{i} - 5\mathbf{j})$$

$$\mathbf{u}_0 = (-6 - 6)\mathbf{i} + (1 + 15)\mathbf{j} = -12\mathbf{i} + 16\mathbf{j}$$

$$u = |\mathbf{u}_0| = \sqrt{144 + 256} = \sqrt{400} = 20$$

$u = 20\ \mathrm{m\,s^{-1}}$

Question 3

Worked Solution

i and j due east and due north. Ships P and Q.

At midnight: P at $(20\mathbf{i}+10\mathbf{j})$ km, Q at $(14\mathbf{i}-6\mathbf{j})$ km.

Three hours later, P is at $(29\mathbf{i}+34\mathbf{j})$ km.

Q travels with velocity $12\mathbf{j}$ km h^{-1}.

Part (a): Velocity of P

$$\mathbf{v}_P = \frac{(29\mathbf{i}+34\mathbf{j})-(20\mathbf{i}+10\mathbf{j})}{3} = \frac{9\mathbf{i}+24\mathbf{j}}{3} = (3\mathbf{i}+8\mathbf{j})\ \text{km h}^{-1}$$

$\mathbf{v}_P = (3\mathbf{i}+8\mathbf{j})\ \text{km h}^{-1}$

Part (b): Expressions for p and q

$\mathbf{p} = (20\mathbf{i}+10\mathbf{j}) + t(3\mathbf{i}+8\mathbf{j}) = (20+3t)\mathbf{i} + (10+8t)\mathbf{j}$
$\mathbf{q} = (14\mathbf{i}-6\mathbf{j}) + 12t\mathbf{j} = 14\mathbf{i} + (-6+12t)\mathbf{j}$

Part (c): Show $d^2 = 25t^2 - 92t + 292$

$$\overrightarrow{PQ} = \mathbf{q}-\mathbf{p} = (14-20-3t)\mathbf{i} + (-6+12t-10-8t)\mathbf{j} = (-6-3t)\mathbf{i} + (4t-16)\mathbf{j}$$

$$d^2 = (-6-3t)^2 + (4t-16)^2 = 36+36t+9t^2+16t^2-128t+256$$
$$= 25t^2 - 92t + 292 \qquad \checkmark$$

Part (d): Time lights on Q move out of sight

Condition: $d \leq 15$ km, i.e. $d^2 = 225$.

$$25t^2 - 92t + 292 = 225 \implies 25t^2 - 92t + 67 = 0$$

$$(t-1)(25t-67) = 0 \implies t = 1 \text{ or } t = \frac{67}{25} = 2.68 \text{ h}$$

At $t = 1$, the lights come into sight. Lights go out of sight at $t = 2.68$ h.

2.68 h = 2 h 40.8 min ≈ 2 h 41 min after midnight = 02:41.

Lights move out of sight at approximately $t = 2.68$ h, i.e. at **02:41**.

Question 4

Worked Solution

i due east, j due north.

Ship S: velocity $(-2.5\mathbf{i} + 6\mathbf{j})$ km h^{-1}. At 1200, position $(16\mathbf{i} + 5\mathbf{j})$ km.

Part (a): Speed of S

$$|\mathbf{v}| = \sqrt{2.5^2 + 6^2} = \sqrt{6.25 + 36} = \sqrt{42.25} = 6.5 \text{ km h}^{-1}$$

Speed $= 6.5$ km h^{-1}

Part (b): Bearing of S

S moves west $(-\mathbf{i})$ and north $(+\mathbf{j})$: northwest quadrant. Bearing measured from north:

$$\tan\phi = \frac{2.5}{6} \implies \phi = \arctan(0.4167) = 22.6^\circ \text{ west of north}$$

Bearing $= 360^\circ - 22.6^\circ = 337^\circ$ (i.e. N 22.6° W)

Part (c): Position of rock R

S hits R at 1500 (3 hours after 1200):

$$\mathbf{R} = (16\mathbf{i} + 5\mathbf{j}) + 3(-2.5\mathbf{i} + 6\mathbf{j}) = (16 - 7.5)\mathbf{i} + (5 + 18)\mathbf{j} = (8.5\mathbf{i} + 23\mathbf{j}) \text{ km}$$

Position of $R = (8.5\mathbf{i} + 23\mathbf{j})$ km

Part (d): Position vector of S at time t hours after 1400

At 1400 (2 hours after 1200):

$$\mathbf{s}_{1400} = (16\mathbf{i} + 5\mathbf{j}) + 2(-2.5\mathbf{i} + 6\mathbf{j}) = (16 - 5)\mathbf{i} + (5 + 12)\mathbf{j} = 11\mathbf{i} + 17\mathbf{j}$$

New velocity from 1400: $5\mathbf{j}$ km h^{-1} (due north at 5 km/h).

$\mathbf{s} = (11\mathbf{i} + 17\mathbf{j}) + 5t\mathbf{j} = 11\mathbf{i} + (17 + 5t)\mathbf{j}$

Part (e): Time when S is due east of R

S is due east of R when $\mathbf{j}$-components are equal:

$$17 + 5t = 23 \implies 5t = 6 \implies t = 1.2 \text{ h after 1400}$$

$t = 1.2$ h after 1400, i.e. at **15:12**

Part (f): Distance from R at 1600

At 1600, $t = 2$ hours after 1400:

$$\mathbf{s}_{1600} = 11\mathbf{i} + (17 + 10)\mathbf{j} = 11\mathbf{i} + 27\mathbf{j}$$

$$\overrightarrow{RS} = (11 - 8.5)\mathbf{i} + (27 - 23)\mathbf{j} = 2.5\mathbf{i} + 4\mathbf{j}$$

$$|\overrightarrow{RS}| = \sqrt{2.5^2 + 4^2} = \sqrt{6.25 + 16} = \sqrt{22.25} \approx 4.72 \text{ km}$$

Distance from R at 1600 ≈ 4.72 km

Question 5

Worked Solution

Particle P, mass 2 kg, constant force $\mathbf{F}$.

At $t = 0$: $\mathbf{v} = (2\mathbf{i} - 5\mathbf{j})\ \mathrm{m\,s^{-1}}$. At $t = 5$: $\mathbf{v} = (7\mathbf{i} + 10\mathbf{j})\ \mathrm{m\,s^{-1}}$.

Part (a): Speed at $t = 0$

$$|\mathbf{v}| = \sqrt{4 + 25} = \sqrt{29} \approx 5.39\ \mathrm{m\,s^{-1}}$$

Speed $= \sqrt{29} \approx 5.39\ \mathrm{m\,s^{-1}}$

Part (b): Find F

$$\mathbf{a} = \frac{\mathbf{v} - \mathbf{u}}{t} = \frac{(7\mathbf{i} + 10\mathbf{j}) - (2\mathbf{i} - 5\mathbf{j})}{5} = \frac{5\mathbf{i} + 15\mathbf{j}}{5} = (\mathbf{i} + 3\mathbf{j})\ \mathrm{m\,s^{-2}}$$
$$\mathbf{F} = m\mathbf{a} = 2(\mathbf{i} + 3\mathbf{j})$$

$\mathbf{F} = (2\mathbf{i} + 6\mathbf{j})$ N

Part (c): Value of t when P moves parallel to i

P moves parallel to $\mathbf{i}$ when the $\mathbf{j}$ component of velocity is zero:

$$\mathbf{v} = (2\mathbf{i} - 5\mathbf{j}) + t(\mathbf{i} + 3\mathbf{j}) = (2 + t)\mathbf{i} + (-5 + 3t)\mathbf{j}$$

$$-5 + 3t = 0 \implies t = \frac{5}{3}$$

$t = \frac{5}{3}$ s

Question 6

Worked Solution

Boat B: at noon, position $(3\mathbf{i} - 4\mathbf{j})$ km. At 1430, position $(8\mathbf{i} + 11\mathbf{j})$ km.

Part (a): Velocity of B

Time elapsed: 2.5 hours.

$$\mathbf{v}_B = \frac{(8\mathbf{i} + 11\mathbf{j}) - (3\mathbf{i} - 4\mathbf{j})}{2.5} = \frac{5\mathbf{i} + 15\mathbf{j}}{2.5} = (2\mathbf{i} + 6\mathbf{j})\ \mathrm{km\,h^{-1}}$$

$\mathbf{v}_B = (2\mathbf{i} + 6\mathbf{j})\ \mathrm{km\,h^{-1}}$

Part (b): Expression for b

$\mathbf{b} = (3\mathbf{i} - 4\mathbf{j}) + t(2\mathbf{i} + 6\mathbf{j}) = (3 + 2t)\mathbf{i} + (-4 + 6t)\mathbf{j}$

Part (c): Find λ such that C intercepts B

C: $\mathbf{c} = (-9\mathbf{i} + 20\mathbf{j}) + t(6\mathbf{i} + \lambda\mathbf{j})$.

For C to intercept B, set $\mathbf{b} = \mathbf{c}$:

$\mathbf{i}$ component: $3 + 2t = -9 + 6t \implies 12 = 4t \implies t = 3$

$\mathbf{j}$ component: $-4 + 6(3) = 20 + 3\lambda \implies 14 = 20 + 3\lambda \implies \lambda = -2$

$\lambda = -2$

Part (d): Show speeds are equal before interception

$|\mathbf{v}_B| = \sqrt{4 + 36} = \sqrt{40}\ \mathrm{km\,h^{-1}}$

$|\mathbf{v}_C| = \sqrt{36 + (-2)^2} = \sqrt{36 + 4} = \sqrt{40}\ \mathrm{km\,h^{-1}}$

Both speeds equal $\sqrt{40}\ \mathrm{km\,h^{-1}}$. ✓

Question 7

Worked Solution

i and j due east and due north. Positions relative to fixed origin O.

Ship S: velocity $(3\mathbf{i} + 3\mathbf{j})$ km h^{-1}. At $t = 0$, position $(-4\mathbf{i} + 2\mathbf{j})$ km.

Ship T: velocity $(-2\mathbf{i} + n\mathbf{j})$ km h^{-1}. At $t = 0$, position $(6\mathbf{i} + \mathbf{j})$ km. Ships meet at P.

Part (a): Position vector of S at time t hours

$$\mathbf{s} = (-4\mathbf{i} + 2\mathbf{j}) + t(3\mathbf{i} + 3\mathbf{j}) = (-4 + 3t)\mathbf{i} + (2 + 3t)\mathbf{j}$$

$\mathbf{s} = (-4 + 3t)\mathbf{i} + (2 + 3t)\mathbf{j}$

Part (b): Find n

Position of T: $\mathbf{t} = (6\mathbf{i} + \mathbf{j}) + t(-2\mathbf{i} + n\mathbf{j}) = (6 - 2t)\mathbf{i} + (1 + nt)\mathbf{j}$.

For meeting: $\mathbf{s} = \mathbf{t}$

$\mathbf{i}$: $-4 + 3t = 6 - 2t \implies 5t = 10 \implies t = 2$

$\mathbf{j}$: $2 + 3(2) = 1 + n(2) \implies 8 = 1 + 2n \implies n = 3.5$

$n = 3.5$

Part (c): Distance OP

At $t = 2$:

$$P = (-4 + 6)\mathbf{i} + (2 + 6)\mathbf{j} = 2\mathbf{i} + 8\mathbf{j}$$
$$OP = \sqrt{4 + 64} = \sqrt{68} = 2\sqrt{17} \approx 8.25 \text{ km}$$

$OP = 2\sqrt{17} \approx 8.25$ km

Question 8

Worked Solution

i due east, j due north.

Velocity of P: $\mathbf{v} = (1-2t)\mathbf{i} + (3t-3)\mathbf{j}$ m s^{-1}.

Part (a): Speed of P when $t = 0$

At $t = 0$: $\mathbf{v} = \mathbf{i} - 3\mathbf{j}$.

$$|\mathbf{v}| = \sqrt{1+9} = \sqrt{10} \approx 3.16 \text{ m s}^{-1}$$

Speed $= \sqrt{10} \approx 3.16$ m s^{-1}

Part (b): Bearing when $t = 2$

At $t = 2$: $\mathbf{v} = (1-4)\mathbf{i} + (6-3)\mathbf{j} = -3\mathbf{i} + 3\mathbf{j}$.

Moving northwest (west 3, north 3 equally). Bearing $= 315°$.

Bearing $= 315°$

Part (c): Values of t

(i) Parallel to $\mathbf{j}$: $\mathbf{i}$ component $= 0$:

$$1 - 2t = 0 \implies t = 0.5 \text{ s}$$

(ii) Parallel to $(-\mathbf{i} - 3\mathbf{j})$: ratio of components $= \frac{-1}{-3} = \frac{1}{3}$:

$$\frac{1-2t}{3t-3} = \frac{-1}{-3} = \frac{1}{3}$$

$$3(1-2t) = 3t - 3 \implies 3 - 6t = 3t - 3 \implies 6 = 9t \implies t = \frac{2}{3} \text{ s}$$

(i) $t = 0.5$ s (ii) $t = \frac{2}{3} \approx 0.67$ s

End of Worked Solutions