Question 1

Worked Solution

i and j are unit vectors due east and due north respectively.

Initial velocity $\mathbf{u} = 0.6\mathbf{j}\ \mathrm{m\,s^{-1}}$ (due north), constant acceleration $\mathbf{a}$.

Part (a): Show that $\mathbf{a} = (0.7\mathbf{i} - 0.1\mathbf{j})\ \mathbf{m\,s^{-2}}$

At $t = 15$, $\mathbf{v} = (10.5\mathbf{i} - 0.9\mathbf{j})\ \mathrm{m\,s^{-1}}$.

Using $\mathbf{v} = \mathbf{u} + \mathbf{a}t$:

$$10.5\mathbf{i} - 0.9\mathbf{j} = 0.6\mathbf{j} + 15\mathbf{a}$$

$$15\mathbf{a} = 10.5\mathbf{i} - 1.5\mathbf{j}$$

$$\mathbf{a} = (0.7\mathbf{i} - 0.1\mathbf{j})\ \mathrm{m\,s^{-2}} \qquad \checkmark$$

Part (b): Find r in terms of t

Using $\mathbf{r} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$ (starting from O at $t = 0$):

$$\mathbf{r} = 0.6t\,\mathbf{j} + \tfrac{1}{2}(0.7\mathbf{i} - 0.1\mathbf{j})t^2 = 0.35t^2\,\mathbf{i} + (0.6t - 0.05t^2)\,\mathbf{j}$$

Part (c): Value of t when boat is north-east of O

North-east of O means the $\mathbf{i}$ and $\mathbf{j}$ components of $\mathbf{r}$ are equal and positive:

$$0.35t^2 = 0.6t - 0.05t^2$$

$$0.4t^2 = 0.6t$$

$$t(0.4t - 0.6) = 0 \implies t = 0 \text{ or } t = \frac{0.6}{0.4} = 1.5$$

$t = 0$ gives the origin; the required solution is:

$t = 1.5$ s

Part (d): Value of t when boat moves in north-east direction

North-east direction means $\mathbf{i}$ and $\mathbf{j}$ components of velocity are equal:

$$\mathbf{v} = (0.6\mathbf{j}) + (0.7\mathbf{i} - 0.1\mathbf{j})t = 0.7t\,\mathbf{i} + (0.6 - 0.1t)\,\mathbf{j}$$

Setting components equal:

$$0.7t = 0.6 - 0.1t \implies 0.8t = 0.6 \implies t = 0.75$$

$t = 0.75$ s

Question 2

Worked Solution

i and j are unit vectors due east and due north. Positions relative to fixed origin O.

$\mathbf{v}_P = (9\mathbf{i} - 2\mathbf{j})\ \mathrm{km\,h^{-1}}$, $\mathbf{v}_Q = (4\mathbf{i} + 8\mathbf{j})\ \mathrm{km\,h^{-1}}$.

Part (a): Bearing of P

P moves with velocity $(9\mathbf{i} - 2\mathbf{j})$: east component 9, south component 2.

$$\tan\theta = \frac{2}{9} \implies \theta = 12.5^\circ \text{ from north (east side)}$$

Bearing $= 90^\circ + \arctan\left(\frac{2}{9}\right) \approx 103^\circ$

Part (b): Expressions for p and q

At $t = 0$: P at $(9\mathbf{i} + 10\mathbf{j})$, Q at $(\mathbf{i} + 4\mathbf{j})$.

(i) $\mathbf{p} = (9\mathbf{i} + 10\mathbf{j}) + t(9\mathbf{i} - 2\mathbf{j}) = (9 + 9t)\mathbf{i} + (10 - 2t)\mathbf{j}$
(ii) $\mathbf{q} = (\mathbf{i} + 4\mathbf{j}) + t(4\mathbf{i} + 8\mathbf{j}) = (1 + 4t)\mathbf{i} + (4 + 8t)\mathbf{j}$

Part (c): Show $\overrightarrow{QP} = (8 + 5t)\mathbf{i} + (6 - 10t)\mathbf{j}$

$$\overrightarrow{QP} = \mathbf{p} - \mathbf{q} = [(9 + 9t) - (1 + 4t)]\,\mathbf{i} + [(10 - 2t) - (4 + 8t)]\,\mathbf{j}$$
$$= (8 + 5t)\,\mathbf{i} + (6 - 10t)\,\mathbf{j} \qquad \checkmark$$

Part (d): Values of t when ships are 10 km apart

$$|\overrightarrow{QP}|^2 = (8 + 5t)^2 + (6 - 10t)^2 = 100$$
$$64 + 80t + 25t^2 + 36 - 120t + 100t^2 = 100$$
$$125t^2 - 40t + 100 = 100$$
$$125t^2 - 40t = 0$$
$$5t(25t - 8) = 0$$

$t = 0$ or $t = \dfrac{8}{25} = 0.32$ h

Question 3

Worked Solution

i and j due east and due north. Positions relative to fixed origin O.

$\mathbf{v}_P = (15\mathbf{i} + 20\mathbf{j})\ \mathrm{m\,s^{-1}}$, $\mathbf{v}_Q = (20\mathbf{i} - 5\mathbf{j})\ \mathrm{m\,s^{-1}}$.

At $t = 0$: P at $400\mathbf{i}$, Q at $800\mathbf{j}$.

Part (a): Bearing of Q

Q moves east 20, south 5.

$$\tan\theta = \frac{5}{20} = \frac{1}{4} \implies \theta = 14.04^\circ \text{ from east} \implies \text{bearing} = 90^\circ + 14.04^\circ = 104^\circ$$

Bearing of $Q \approx 104^\circ$

Part (b): Expressions for p and q

(i) $\mathbf{p} = 400\mathbf{i} + t(15\mathbf{i} + 20\mathbf{j}) = (400 + 15t)\mathbf{i} + 20t\,\mathbf{j}$
(ii) $\mathbf{q} = 800\mathbf{j} + t(20\mathbf{i} - 5\mathbf{j}) = 20t\,\mathbf{i} + (800 - 5t)\,\mathbf{j}$

Part (c): Position vector of Q when Q is due west of P

Q is due west of P when the $\mathbf{j}$ (north) components are equal:

$$20t = 800 - 5t \implies 25t = 800 \implies t = 32 \text{ s}$$

At $t = 32$:

$$\mathbf{q} = 20(32)\,\mathbf{i} + (800 - 5 \times 32)\,\mathbf{j} = 640\mathbf{i} + 640\mathbf{j}$$

Position vector of $Q = (640\mathbf{i} + 640\mathbf{j})$ m

Question 4

Worked Solution

$\mathbf{F}_1 = (-\mathbf{i} + 2\mathbf{j})$ N. $\mathbf{F}_2 = k(\mathbf{i} + \mathbf{j})$ for some $k > 0$ (acts in direction $\mathbf{i} + \mathbf{j}$).

Part (a): Find $\mathbf{F}_2$

Resultant $\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 = (-1 + k)\mathbf{i} + (2 + k)\mathbf{j}$.

$\mathbf{R}$ acts in direction $(\mathbf{i} + 3\mathbf{j})$, so the ratio of components equals 1 : 3:

$$\frac{-1 + k}{2 + k} = \frac{1}{3}$$

$$3(-1 + k) = 2 + k \implies -3 + 3k = 2 + k \implies 2k = 5 \implies k = 2.5$$

$\mathbf{F}_2 = 2.5\mathbf{i} + 2.5\mathbf{j}$ N

Part (b): Speed of P when $t = 3$ s

Acceleration $\mathbf{a} = (3\mathbf{i} + 9\mathbf{j})$ m s^{-2}. At $t = 0$, $\mathbf{v}_0 = (3\mathbf{i} - 22\mathbf{j})$ m s^{-1}.

$$\mathbf{v} = (3\mathbf{i} - 22\mathbf{j}) + 3(3\mathbf{i} + 9\mathbf{j}) = (3 + 9)\mathbf{i} + (-22 + 27)\mathbf{j} = 12\mathbf{i} + 5\mathbf{j}$$
$$|\mathbf{v}| = \sqrt{12^2 + 5^2} = \sqrt{144 + 25} = \sqrt{169} = 13$$

Speed $= 13$ m s^{-1}

Question 5

Worked Solution

Mass $m = 0.5$ kg, force $\mathbf{F} = (3\mathbf{i} - 2\mathbf{j})$ N.

Part (a): Show that $|\mathbf{a}| = 2\sqrt{13}$ m s^{-2}

$\mathbf{F} = m\mathbf{a}$:

$$\mathbf{a} = \frac{\mathbf{F}}{m} = \frac{(3\mathbf{i} - 2\mathbf{j})}{0.5} = (6\mathbf{i} - 4\mathbf{j})\ \text{m s}^{-2}$$

$$|\mathbf{a}| = \sqrt{6^2 + (-4)^2} = \sqrt{36 + 16} = \sqrt{52} = 2\sqrt{13}\ \text{m s}^{-2} \qquad \checkmark$$

Part (b): Velocity of P at $t = 2$ s

$\mathbf{u} = (\mathbf{i} + 3\mathbf{j})$ m s^{-1}.

$$\mathbf{v} = (\mathbf{i} + 3\mathbf{j}) + 2(6\mathbf{i} - 4\mathbf{j}) = (1 + 12)\mathbf{i} + (3 - 8)\mathbf{j}$$

$\mathbf{v} = (13\mathbf{i} - 5\mathbf{j})$ m s^{-1}

Part (c): Distance moved by Q in 2 s

$\mathbf{v}_Q = (2\mathbf{i} - \mathbf{j})$ m s^{-1} (constant).

Speed $= \sqrt{4 + 1} = \sqrt{5}$. Distance $= 2\sqrt{5}$.

Distance $= 2\sqrt{5} \approx 4.47$ m

Part (d): Show both particles move in same direction at $t = 3.5$ s

Velocity of P at $t = 3.5$:

$$\mathbf{v}_P = (\mathbf{i} + 3\mathbf{j}) + 3.5(6\mathbf{i} - 4\mathbf{j}) = (1 + 21)\mathbf{i} + (3 - 14)\mathbf{j} = 22\mathbf{i} - 11\mathbf{j}$$

This is $11(2\mathbf{i} - \mathbf{j})$, which is a positive scalar multiple of $\mathbf{v}_Q = (2\mathbf{i} - \mathbf{j})$.

At $t = 3.5$: $\mathbf{v}_P = 11(2\mathbf{i} - \mathbf{j}) = 11\,\mathbf{v}_Q$. Since this is a positive multiple of $\mathbf{v}_Q$, both particles are moving in the same direction. ✓

Question 6

Worked Solution

Two forces $(4\mathbf{i} - 2\mathbf{j})$ N and $(2\mathbf{i} + q\mathbf{j})$ N act on P, mass 1.5 kg. Resultant is parallel to $(2\mathbf{i} + \mathbf{j})$.

Part (a): Find q

Resultant $= (6\mathbf{i} + (q-2)\mathbf{j})$. Parallel to $(2\mathbf{i} + \mathbf{j})$ means:

$$\frac{q-2}{6} = \frac{1}{2} \implies q - 2 = 3 \implies q = 5$$

$q = 5$

Part (b): Speed of P at $t = 2$ s

Resultant force $= (6\mathbf{i} + 3\mathbf{j})$ N.

$$\mathbf{a} = \frac{(6\mathbf{i} + 3\mathbf{j})}{1.5} = (4\mathbf{i} + 2\mathbf{j})\ \mathrm{m\,s^{-2}}$$

At $t = 0$, $\mathbf{v}_0 = (-2\mathbf{i} + 4\mathbf{j})\ \mathrm{m\,s^{-1}}$.

$$\mathbf{v} = (-2\mathbf{i} + 4\mathbf{j}) + 2(4\mathbf{i} + 2\mathbf{j}) = 6\mathbf{i} + 8\mathbf{j}$$
$$|\mathbf{v}| = \sqrt{36 + 64} = \sqrt{100} = 10$$

Speed $= 10\ \mathrm{m\,s^{-1}}$

Question 7

Worked Solution

Acceleration $\mathbf{a} = (2\mathbf{i} - 5\mathbf{j})\ \mathrm{m\,s^{-2}}$. At $t = 0$, speed $= u$. At $t = 3$, $\mathbf{v} = (-6\mathbf{i} + \mathbf{j})\ \mathrm{m\,s^{-1}}$.

Using $\mathbf{v} = \mathbf{u}_0 + \mathbf{a}t$:

$$-6\mathbf{i} + \mathbf{j} = \mathbf{u}_0 + 3(2\mathbf{i} - 5\mathbf{j})$$

$$\mathbf{u}_0 = (-6 - 6)\mathbf{i} + (1 + 15)\mathbf{j} = -12\mathbf{i} + 16\mathbf{j}$$

$$u = |\mathbf{u}_0| = \sqrt{144 + 256} = \sqrt{400} = 20$$

$u = 20\ \mathrm{m\,s^{-1}}$

Question 8

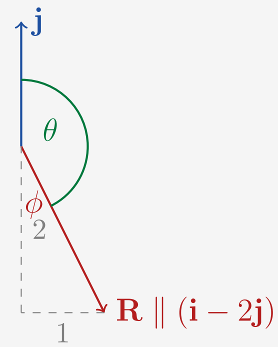
Worked Solution

Two forces $(4\mathbf{i} - 5\mathbf{j})$ N and $(p\mathbf{i} + q\mathbf{j})$ N on particle P, mass m kg. Resultant $\mathbf{R}$ parallel to $(\mathbf{i} - 2\mathbf{j})$.

$\mathbf{R} = (4 + p)\mathbf{i} + (q - 5)\mathbf{j}$.

Part (a): Angle between R and j

$\mathbf{R}$ is parallel to $(\mathbf{i} - 2\mathbf{j})$. The diagram below shows $\mathbf{R}$ (in the direction $\mathbf{i} - 2\mathbf{j}$) and $\mathbf{j}$, with the angle θ between them.



From the diagram, $\mathbf{R}$ points into the fourth quadrant (positive $\mathbf{i}$, negative $\mathbf{j}$ components). The acute angle ϕ between $\mathbf{R}$ and the negative $\mathbf{j}$-axis satisfies:

$$\tan\phi = \frac{1}{2} \implies \phi = \arctan\left(\tfrac{1}{2}\right) = 26.57^\circ$$

Since $\mathbf{R}$ lies on the opposite side of the origin to $\mathbf{j}$, the angle θ measured from $\mathbf{j}$ to $\mathbf{R}$ is:

$$\theta = 180^\circ - 26.57^\circ = 153.4^\circ$$

Angle between $\mathbf{R}$ and $\mathbf{j} = 153.4^\circ$

Part (b): Show that $2p + q + 3 = 0$

$\mathbf{R}$ parallel to $(\mathbf{i} - 2\mathbf{j})$ means:

$$\frac{q-5}{4+p} = \frac{-2}{1} \implies q - 5 = -2(4 + p) = -8 - 2p$$

$$q - 5 + 8 + 2p = 0 \implies 2p + q + 3 = 0 \qquad \checkmark$$

Part (c): Find m given $q = 1$, $|\mathbf{a}| = 8\sqrt{5}$ m s^{-2}

With $q = 1$: $2p + 1 + 3 = 0 \implies p = -2$.

$$\mathbf{R} = (4 - 2)\mathbf{i} + (1 - 5)\mathbf{j} = 2\mathbf{i} - 4\mathbf{j}$$

$$|\mathbf{R}| = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$$

Using $\mathbf{F} = m\mathbf{a}$:

$$2\sqrt{5} = m \times 8\sqrt{5} \implies m = \frac{2\sqrt{5}}{8\sqrt{5}} = \frac{1}{4}$$

$m = \dfrac{1}{4}$ kg

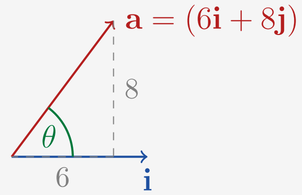
Question 9

Worked Solution

Mass $m = 0.4$ kg, acceleration $\mathbf{a} = (6\mathbf{i} + 8\mathbf{j})\ \mathrm{m\,s^{-2}}$.

Part (a): Angle between acceleration and i

The diagram below shows $\mathbf{a} = (6\mathbf{i} + 8\mathbf{j})$ and the unit vector $\mathbf{i}$, with θ the angle between them.



$$\tan\theta = \frac{8}{6} = \frac{4}{3} \implies \theta = \arctan\left(\tfrac{4}{3}\right) \approx 53^\circ$$

Angle between $\mathbf{a}$ and $\mathbf{i} \approx 53^\circ$

Part (b): Magnitude of F

$$\mathbf{F} = m\mathbf{a} = 0.4(6\mathbf{i} + 8\mathbf{j}) = (2.4\mathbf{i} + 3.2\mathbf{j})\ \mathrm{N}$$
$$|\mathbf{F}| = \sqrt{2.4^2 + 3.2^2} = \sqrt{5.76 + 10.24} = \sqrt{16} = 4\ \mathrm{N}$$

$|\mathbf{F}| = 4$ N

Part (c): Velocity of P when $t = 5$

$\mathbf{v}_0 = 9\mathbf{i} - 10\mathbf{j}\ \mathrm{m\,s^{-1}}$.

$$\mathbf{v} = (9\mathbf{i} - 10\mathbf{j}) + 5(6\mathbf{i} + 8\mathbf{j}) = (9 + 30)\mathbf{i} + (-10 + 40)\mathbf{j}$$

$\mathbf{v} = (39\mathbf{i} + 30\mathbf{j})\ \mathrm{m\,s^{-1}}$

End of Worked Solutions