Question 1

Worked Solution

Distribution of S: $P(S = 0) = a$, $P(S = 1) = b$, $P(S = 2) = c$, $P(S = 3) = 0.1$, $P(S = 4) = 0.15$.

Step 1: Set up equations.

Sum of all probabilities $= 1$:

$$a + b + c + 0.1 + 0.15 = 1 \implies a + b + c = 0.75 \quad (1)$$

"Probability of scoring less than 2 is twice the probability of scoring at least 2":

$$P(S < 2) = 2\,P(S \geq 2)$$

$$(a + b) = 2(c + 0.1 + 0.15) \implies a + b = 2c + 0.5 \quad (2)$$

Step 2: Solve for c.

From (1): $a + b = 0.75 - c$. Substitute into (2):

$$0.75 - c = 2c + 0.5 \implies 3c = 0.25 \implies c = \frac{1}{12} \approx 0.0833$$

So $a + b = 0.75 - \frac{1}{12} = \frac{9}{12} - \frac{1}{12} = \frac{8}{12} = \frac{2}{3}$.

Note: the individual values of a and b are not uniquely determined by the given information; only c and $a + b$ are fixed. The probability we need only requires c and $a + b$.

Step 3: $P(\text{total} = 6)$ when John plays twice.

The pairs (S_1, S_2) that give a total of 6:

- $(2, 4)$ and $(4, 2)$: probability $= 2 \times c \times 0.15 = 2 \times \frac{1}{12} \times 0.15$
- $(3, 3)$: probability $= 0.1 \times 0.1 = 0.01$

$$P(\text{total} = 6) = 2 \times \frac{1}{12} \times 0.15 + (0.1)^2 = \frac{0.30}{12} + 0.01 = 0.025 + 0.01 = 0.035$$

$$P(\text{total} = 6) = 0.035 \qquad \left(\text{or } \frac{7}{200}\right)$$

Question 2

Worked Solution

D takes values 10, 20, 30, 40, 50 with probabilities $\frac{k}{10}, \frac{k}{20}, \frac{k}{30}, \frac{k}{40}, \frac{k}{50}$.

Part (a): Show $k = \frac{600}{137}$

Sum of probabilities = 1:

$$\frac{k}{10} + \frac{k}{20} + \frac{k}{30} + \frac{k}{40} + \frac{k}{50} = 1$$

Finding common denominator 600:

$$k\left(\frac{60+30+20+15+12}{600}\right) = 1 \implies k \times \frac{137}{600} = 1 \implies k = \frac{600}{137} \quad \checkmark$$

$$k = \frac{600}{137} \checkmark$$

Part (b): $\mathrm{P}(D_1 + D_2 = 80)$ where D_1, D_2 are independent copies of D

The pairs (D_1, D_2) summing to 80:

- $(30, 50)$ and $(50, 30)$: each with probability $\frac{k}{30} \times \frac{k}{50}$
- $(40, 40)$: probability $\left(\frac{k}{40}\right)^2$

$$\mathrm{P}(D_1 + D_2 = 80) = 2 \times \frac{k}{30} \times \frac{k}{50} + \left(\frac{k}{40}\right)^2 = \frac{2k^2}{1500} + \frac{k^2}{1600}$$

Using $k = \frac{600}{137}$, so $k^2 = \frac{360000}{18769}$:

$$= k^2\left(\frac{2}{1500} + \frac{1}{1600}\right) = \frac{360000}{18769} \times \left(\frac{2}{1500} + \frac{1}{1600}\right)$$

$$= \frac{360000}{18769} \times \frac{3200+1500}{2{,}400{,}000} = \frac{360000 \times 4700}{18769 \times 2{,}400{,}000} = \frac{4700}{18769 \times \frac{20}{3}} = 0.03756\ldots$$

$$\mathrm{P}(D_1 + D_2 = 80) = 0.0376 \quad \text{(3 s.f.)}$$

Part (c): Smallest angle of quadrilateral Q is more than 50

The four angles of quadrilateral Q are the first four terms of an arithmetic sequence with common difference d (a single observation of D). Let the first term be a.

The four angles are: $a,\ a + d,\ a + 2d,\ a + 3d$.

Sum of angles in a quadrilateral = 360:

$$4a + 6d = 360 \implies 2a + 3d = 180 \implies a = \frac{180 - 3d}{2} = 90 - \frac{3d}{2}$$

The **smallest angle** is a (the first term, since $d > 0$). We need $a > 50$:

$$90 - \frac{3d}{2} > 50 \implies \frac{3d}{2} < 40 \implies d < \frac{80}{3} = 26.\overline{6}$$

Since D takes values 10, 20, 30, 40, 50, the values satisfying $d < 26.\overline{6}$ are $d = 10$ and $d = 20$.

But we must also check that all four angles are positive (valid angles). For $d = 30$: $a = 90 - 45 = 45 > 0$, but $a \not> 50$. For $d = 10$: $a = 90 - 15 = 75 > 50$. For $d = 20$: $a = 90 - 30 = 60 > 50$.

$$\text{P}(D = 10 \text{ or } D = 20) = \frac{k}{10} + \frac{k}{20} = k\left(\frac{1}{10} + \frac{1}{20}\right) = k \times \frac{3}{20} = \frac{600}{137} \times \frac{3}{20} = \frac{1800}{2740} = \frac{90}{137}$$

$$\text{P(smallest angle} > 50) = \frac{90}{137}$$

Question 3

Worked Solution

Distribution of X: values $-4, -2, 1, 3, 5$ with probabilities $0.4, p, 0.05, 0.15, p$.

Part (a): Show $p = 0.2$

$$0.4 + p + 0.05 + 0.15 + p = 1 \implies 2p + 0.6 = 1 \implies 2p = 0.4 \implies p = 0.2 \quad \checkmark$$

$p = 0.2$ ✓

Part (b): $\mathrm{E}(X)$

$$\mathrm{E}(X) = \sum x \cdot \mathrm{P}(X = x)$$
$$= (-4)(0.4) + (-2)(0.2) + (1)(0.05) + (3)(0.15) + (5)(0.2)$$
$$= -1.6 - 0.4 + 0.05 + 0.45 + 1.0 = -0.5$$

$\mathrm{E}(X) = -0.5$

Part (c): $F(0) = \mathrm{P}(X \leq 0)$

$$F(0) = \mathrm{P}(X \leq 0) = \mathrm{P}(X = -4) + \mathrm{P}(X = -2) = 0.4 + 0.2 = 0.6$$

$F(0) = 0.6$

Part (d): $\mathrm{P}(3X + 2 > 5)$

$$3X + 2 > 5 \implies 3X > 3 \implies X > 1$$

So we need $X = 3$ or $X = 5$:

$$\mathrm{P}(X > 1) = \mathrm{P}(X = 3) + \mathrm{P}(X = 5) = 0.15 + 0.2 = 0.35$$

$\mathrm{P}(3X + 2 > 5) = 0.35$

Part (e): Find a **such that** $\mathrm{Var}(aX + 3) = 53.4$**, given** $\mathrm{Var}(X) = 13.35$

Adding a constant does not change the variance; multiplying by a scales variance by a^2:

$$\mathrm{Var}(aX + 3) = a^2\,\mathrm{Var}(X) = 13.35a^2$$

Set equal to 53.4:

$$13.35a^2 = 53.4 \implies a^2 = 4 \implies a = \pm 2$$

$a = 2$ or $a = -2$

Question 4

Worked Solution

X takes values 1, 2, 3, 4. Condition: $\text{P}(X = r) = \text{P}(X = r + 2)$ for $r = 1, 2$, so $\text{P}(X = 1) = \text{P}(X = 3)$ and $\text{P}(X = 2) = \text{P}(X = 4)$. Given $\text{P}(X = 2) = 0.35$.

Part (a): Complete probability distribution

$\text{P}(X = 4) = 0.35$ (from the symmetry condition).

$\text{P}(X = 1) + \text{P}(X = 3) = 1 - 0.35 - 0.35 = 0.30$, and since they're equal: $\text{P}(X = 1) = \text{P}(X = 3) = 0.15$.

x	1	2	3	4
$\text{P}(X = x)$	0.15	0.35	0.15	0.35

$\text{P}(X = 1) = 0.15$, $\text{P}(X = 2) = 0.35$, $\text{P}(X = 3) = 0.15$, $\text{P}(X = 4) = 0.35$

Part (b): 60 spins; P(more than half land on 4)

Let A = number of spins landing on 4. $A \sim \text{B}(60, 0.35)$.

More than half of 60 means $A > 30$:

$$\text{P}(A > 30) = 1 - \text{P}(A \leq 30) = 1 - 0.99411\ldots = 0.00589\ldots$$

$\text{P}(A > 30) = 0.00589 \quad (3 \text{ s.f.})$

Part (c): $Y = \frac{12}{X}$; find $\text{P}(Y - X \leq 4)$

First, find the distribution of Y:

x	1	2	3	4
$y = 12/x$	12	6	4	3
$Y - X$	$12 - 1 = 11$	$6 - 2 = 4$	$4 - 3 = 1$	$3 - 4 = -1$

$Y - X \leq 4$ when $X = 2$, $X = 3$, or $X = 4$ (giving $Y - X = 4, 1, -1$ respectively).

$$\text{P}(Y - X \leq 4) = \text{P}(X = 2) + \text{P}(X = 3) + \text{P}(X = 4) = 0.35 + 0.15 + 0.35 = 0.85$$

$\text{P}(Y - X \leq 4) = 0.85$

Question 5

Worked Solution

X takes values a, b, c with probabilities $\log_{36} a$, $\log_{36} b$, $\log_{36} c$ respectively, where $a < b < c$ are distinct integers, all probabilities > 0.

Part (a): Find a, b, c

Step 1: Sum of probabilities $= 1$:

$$\log_{36} a + \log_{36} b + \log_{36} c = 1$$

Using the log addition rule: $\log_{36}(abc) = 1 \implies abc = 36$.

Step 2: All probabilities must be > 0, so $\log_{36} a > 0 \implies a > 1$. Similarly $b > 1$ and $c > 1$. So a, b, c are distinct integers all greater than 1.

Step 3: We need three distinct integers, each > 1, whose product is 36.

$36 = 2 \times 2 \times 9 = 2 \times 3 \times 6 = 2 \times 18 = 3 \times 12 = 4 \times 9 = \dots$

Since $a < b < c$ and all must be distinct integers > 1: $36 = 2 \times 3 \times 6$ is the only factorisation into three distinct integers all greater than 1.

Check: $\log_{36} 2 + \log_{36} 3 + \log_{36} 6 = \log_{36}(36) = 1$ ✓

$a = 2, \quad b = 3, \quad c = 6$

Part (b): $\mathrm{P}(X_1 = X_2)$ for independent X_1, X_2 with same distribution

$$\mathrm{P}(X_1 = X_2) = (\log_{36} 2)^2 + (\log_{36} 3)^2 + (\log_{36} 6)^2$$

Using $\log_{36} 6 = \frac{1}{2}$ (since $36^{1/2} = 6$), $\log_{36} 2 = \frac{\log 2}{\log 36}$, $\log_{36} 3 = \frac{\log 3}{\log 36}$:

$$= \left(\frac{\log 2}{\log 36}\right)^2 + \left(\frac{\log 3}{\log 36}\right)^2 + \left(\frac{1}{2}\right)^2$$

$$= (0.19332\dots)^2 + (0.30624\dots)^2 + (0.5)^2$$

$$= 0.03737 + 0.09379 + 0.25 = 0.38116\dots$$

$\mathrm{P}(X_1 = X_2) = 0.381 \quad (3 \text{ s.f.}) \quad \left[= (\log_{36} 2)^2 + (\log_{36} 3)^2 + \frac{1}{4}\right]$

End of Worked Solutions