Question 1

Worked Solution

Manufacturer claims 30% of chocolates have hard centres. Sample of 8: 5 have hard centres. Test at 5% whether proportion is not 30%.

Step 1: Hypotheses.

$$H_0 : p = 0.3 \qquad H_1 : p \neq 0.3$$

where p is the proportion of chocolates with hard centres.

Step 2: Under H_0: $X \sim \mathrm{B}(8, 0.3)$. Observed $x = 5$ (upper tail, since $5 > np = 2.4$).

Step 3: Find the tail probability.

$$\mathrm{P}(X \geq 5) = 1 - \mathrm{P}(X \leq 4) = 1 - 0.9420 = 0.0580$$

Also check: $\mathrm{P}(X \geq 6) = 1 - \mathrm{P}(X \leq 5) = 1 - 0.9887 = 0.0113$.

Step 4: Compare with $\alpha/2 = 0.025$.

$\mathrm{P}(X \geq 5) = 0.0580 > 0.025$, so the upper critical region is $X \geq 6$ (where probability drops below 0.025). The critical value is 6.

Since $x = 5 < 6$, the observed value does not lie in the critical region.

Step 5: Conclusion.

Do not reject H_0. There is insufficient evidence at the 5% significance level that the proportion of chocolates with hard centres differs from 30%.

Question 2

Worked Solution

Part (i): $R \sim \mathrm{B}(6, p)$, test $H_0 : p = 0.45$ vs $H_1 : p \neq 0.45$ at 5%, observed $r = 6$

Step 1: Under H_0: $R \sim \mathrm{B}(6, 0.45)$. Observed $r = 6$, in the upper tail.

$$\mathrm{P}(R \geq 6) = \mathrm{P}(R = 6) = (0.45)^6 = 0.00830\ldots \approx 0.0083$$

Step 2: For a two-tailed test at 5%, compare with 0.025: $0.0083 < 0.025$.

The observed value lies in the critical region.

> $\mathrm{P}(R = 6) = 0.0083 < 0.025$. Reject H_0. There is sufficient evidence at the 5% level that $p \neq 0.45$.

Part (ii): $S \sim \mathrm{B}(n, 0.45)$, observed $s = 1$. Find largest n such that H_0 is not rejected.

Same hypotheses. $s = 1$ is in the lower tail. We need $\mathrm{P}(S \leq 1) > 0.025$:

$$\mathrm{P}(S \leq 1) = (0.55)^n + n(0.45)(0.55)^{n-1} > 0.025$$

Test values:

$$\begin{aligned} n = 9 : &\quad (0.55)^9 + 9(0.45)(0.55)^8 = 0.00751 + 0.02478 = 0.0385 > 0.025 \\ n = 10 : &\quad (0.55)^{10} + 10(0.45)(0.55)^9 = 0.00253 + 0.02077 = 0.0233 < 0.025 \end{aligned}$$

At $n = 9$: $\mathrm{P}(S \leq 1) = 0.0385 > 0.025$ — do not reject. At $n = 10$: $\mathrm{P}(S \leq 1) = 0.0233 < 0.025$ — reject.

> Largest n for which H_0 is not rejected is $n = \mathbf{9}$.

Question 3

Worked Solution

Part (i): $n = 14$, $x = 2$, test at 2.5% whether proportion of households receiving Channel C is less than 0.35

Step 1: Hypotheses.

$$H_0 : p = 0.35 \qquad H_1 : p < 0.35$$

Step 2: Under H_0: $X \sim \mathrm{B}(14, 0.35)$. Observed $x = 2$.

$$\begin{aligned} \mathrm{P}(X \le 2) &= (0.65)^{14} + 14(0.35)(0.65)^{13} + \binom{14}{2}(0.35)^2(0.65)^{12} \\ &= 0.00178 + 0.01343 + 0.04696 = 0.06217\ldots \end{aligned}$$

Check: $\mathrm{P}(X \le 1) = 0.00178 + 0.01343 = 0.01521 < 0.025$.

So critical region is $X \le 1$ (where $\mathrm{P}(X \le 1) = 0.0152 < 0.025$). Observed $x = 2$ does not lie in the critical region.

> $\mathrm{P}(X \le 2) = 0.0622 > 0.025$. Do not reject H_0. Insufficient evidence at 2.5% that the proportion of households receiving Channel C is less than 0.35.

Part (ii): New sample of size n, 0 receive Channel C. Find largest n for H_0 not rejected.

Same hypotheses and significance level. Need $\mathrm{P}(X = 0) = (0.65)^n > 0.025$:

$$n \ln(0.65) > \ln(0.025) \implies n < \frac{\ln(0.025)}{\ln(0.65)} = \frac{-3.6889}{-0.4308} = 8.563\ldots$$

So largest integer is $n = 8$. Verify: $(0.65)^8 = 0.0319 > 0.025$ ✓; $(0.65)^9 = 0.0207 < 0.025$.

> Largest $n = \mathbf{8}$: $(0.65)^8 = 0.0319 > 0.025$, so H_0 not rejected; $(0.65)^9 = 0.0207 < 0.025$, so H_0 would be rejected.

Question 4

Worked Solution

20-letter encoded message contains 1 letter e. In German, letter e appears 19% of the time.

Part (i): Test at 10% whether proportion of e is less than 19%

Step 1: Hypotheses.

$$H_0 : p = 0.19 \qquad H_1 : p < 0.19$$

where p is the probability any given letter in the message is e.

Step 2: Under H_0: $X \sim \mathrm{B}(20, 0.19)$. Observed $x = 1$.

$$\begin{aligned} \mathrm{P}(X \leq 1) &= (0.81)^{20} + 20(0.19)(0.81)^{19} \\ &= 0.01480 + 0.06943 = 0.08423\ldots \end{aligned}$$

Step 3: $\mathrm{P}(X \leq 1) = 0.0842 < 0.10$.

Also: $\mathrm{P}(X = 0) = 0.0148 < 0.10$, so if $x = 0$ it would also be in the critical region. The critical region is $X \leq 1$.

Since $x = 1$ lies in the critical region:

Reject H_0. There is significant evidence at the 10% level that the proportion of letter e is less than 0.19 (less than in German). The language from which the message is a sample uses the letter e less frequently than German.

Part (ii): Why binomial might not be appropriate

The letters in the message are **not independent** — in any language, consecutive letters depend on each other (the probability of one letter depends on preceding letters). Also, the probability p may not be constant from letter to letter within a word.

Question 5

Worked Solution

Agency claims at least 65% of local population favour development. Survey of 12: 6 in favour.

Part (i): Test at 10% whether result is consistent with agency claim

Step 1: Hypotheses.

$$H_0 : p = 0.65 \qquad H_1 : p < 0.65$$

Step 2: Under H_0: $X \sim \mathrm{B}(12, 0.65)$. Observed $x = 6$.

$$\mathrm{P}(X \leq 6) = \sum_{k=0}^{6} \binom{12}{k} (0.65)^k (0.35)^{12-k} = 0.2127$$

Step 3: $\mathrm{P}(X \leq 6) = 0.2127 > 0.10$.

Critical region: $\mathrm{P}(X \leq 5) = 0.0846 < 0.10$, so critical region is $X \leq 5$. Observed $x = 6$ not in critical region.

> Do not reject H_0. Insufficient evidence at 10% that the proportion is less than 65%. The result is consistent with the agency's claim.

Part (ii): Residents' group claim: no more than 35% in favour. Comment.

The residents' claim ($p \leq 0.35$) is even further from $H_0 : p = 0.65$ than the alternative tested. Since we could not even reject H_0 against $H_1 : p < 0.65$, there is certainly insufficient evidence to support $p \leq 0.35$.

> There is insufficient evidence to support the residents' claim. Since the test failed to reject $H_0 : p = 0.65$, there is even less reason to conclude $p \leq 0.35$.

Part (iii): Test at 15%, sample size $2n$, exactly n favour; find smallest n to reject agency's claim

$H_0 : p = 0.65$, $H_1 : p < 0.65$. Observed $x = n$ from $\mathrm{B}(2n, 0.65)$. Need $\mathrm{P}(X \leq n) < 0.15$.

Test:

$$n = 6 : \mathrm{P}(X \leq 6 \mid \mathrm{B}(12, 0.65)) = 0.2127 > 0.15 \quad \text{(don't reject)}$$
$$n = 7 : \mathrm{P}(X \leq 7 \mid \mathrm{B}(14, 0.65)) \approx 0.0898 < 0.15 \quad \text{(reject)}$$

> Smallest $n = \mathbf{7}$: with sample size $2n = 14$ and $x = 7$, $\mathrm{P}(X \leq 7) = 0.0898 < 0.15$, so H_0 is rejected.

Question 6

Worked Solution

Drug development: continue if $p > 0.7$. 14 patients tested. $H_0 : p = 0.7$, $H_1 : p > 0.7$, at 5%.

Part (i): Find critical region

Under H_0: $X \sim \mathrm{B}(14, 0.7)$.

Find smallest c such that $\mathrm{P}(X \geq c) < 0.05$:

$$\begin{aligned}\mathrm{P}(X \geq 13) &= \binom{14}{13}(0.7)^{13}(0.3)^1 + (0.7)^{14} \\ &= 14 \times 0.02542 \times 0.3 + 0.01178 \\ &= 0.10677 + 0.01178 = 0.04754\ldots < 0.05 \quad \checkmark\end{aligned}$$

Check: $\mathrm{P}(X \geq 12) = 0.1608 > 0.05$.

Critical region: $X \geq 13$, with tail probability 0.0475.

Critical region: $\{X \geq 13\}$, with probability 0.0475.

Part (ii): 12 of 14 show improvement; carry out test

$x = 12$ does not lie in the critical region $\{X \geq 13\}$.

$x = 12$ does not lie in the critical region. Do not reject H_0. Insufficient evidence at 5% that the proportion showing improvement is greater than 0.7.

Question 7

Worked Solution

City: 40% from ethnic group Z. Company sample of 12 employees: 2 from group Z. Test at 5% whether company proportion is less.

Part (i): Assumption for validity

The sample must be **random** — each employee equally likely to be selected (random sample from the company).

Part (ii): Appropriate sampling method

List all employees and number them sequentially. Use random numbers to select 12 (simple random sampling).

Part (iii): Carry out the test at 5%

Step 1: Hypotheses.

$$H_0 : p = 0.4 \qquad H_1 : p < 0.4$$

Step 2: Under H_0: $X \sim \mathrm{B}(12, 0.4)$. Observed $x = 2$.

$$\mathrm{P}(X \leq 2) = (0.6)^{12} + 12(0.4)(0.6)^{11} + \binom{12}{2}(0.4)^2(0.6)^{10}$$
$$= 0.00217 + 0.01741 + 0.06385 = 0.08343\ldots$$

$\mathrm{P}(X \leq 1) = 0.00217 + 0.01741 = 0.01958 < 0.05$.

Critical region: $X \leq 1$. Observed $x = 2$ does not lie in the critical region ($\mathrm{P}(X \leq 2) = 0.0834 > 0.05$).

$\mathrm{P}(X \leq 2) = 0.0834 > 0.05$. Do not reject H_0. Insufficient evidence at the 5% level that the proportion of employees from ethnic group Z is less than in the city.

Critical region: $\{X \leq 1\}$, probability 0.0196.

Part (iv): Would a 10% test be more or less supportive of the manager's belief?

At 10%: $\mathrm{P}(X \leq 2) = 0.0834 < 0.10$, so critical region would be $X \leq 2$ and $x = 2$ would lie in the critical region — H_0 would be rejected.

More supportive: at 10% significance, H_0 would be rejected (since $x = 2$ falls in the critical region $X \leq 2$), providing evidence that the company discriminates against ethnic group Z.

Question 8

Worked Solution

55% of pupils are girls. Claim: probability a girl becomes Head Student is greater than 0.55. 6 of last 8 Head Students are girls.

Part (i): Test at 10%

Step 1: Hypotheses.

$$H_0 : p = 0.55 \qquad H_1 : p > 0.55$$

where p is the probability a Head Student is a girl.

Step 2: Under H_0: $R \sim \mathrm{B}(8, 0.55)$. Observed $r = 6$.

$$\mathrm{P}(R \geq 6) = 1 - \mathrm{P}(R \leq 5) = 1 - 0.7799 = 0.2201$$

Step 3: Find critical region: $\mathrm{P}(R \geq 7) = 1 - \mathrm{P}(R \leq 6) = 1 - 0.9368 = 0.0632 < 0.10$.

Critical region: $R \geq 7$, probability 0.0632. Observed $r = 6$ does not lie in the critical region.

$\mathrm{P}(R \geq 6) = 0.2201 > 0.10$. Do not reject H_0. There is insufficient evidence at the 10% significance level that the probability of a girl becoming Head Student is greater than 0.55.

Critical region: $\{R \geq 7\}$, probability 0.0632.

Part (ii): What needs to be assumed about the data for the test to be valid

The last 8 **years** (in which a Head Student was chosen) must form a **random sample** of years — the outcomes must be independent, with each year equally likely to yield a girl Head Student with probability p.

End of Worked Solutions