Question 1

The equation $x^2 + (k-3)x + (3-2k) = 0$ has two distinct real roots. (a) Show that $k^2 + 2k - 3 > 0$. (b) Find the set of possible values of k.

Worked Solution

Part (a):

For two distinct real roots, $b^2 - 4ac > 0$. Here $a = 1$, $b = k-3$, $c = 3-2k$:

$$(k-3)^2 - 4(3-2k) > 0$$

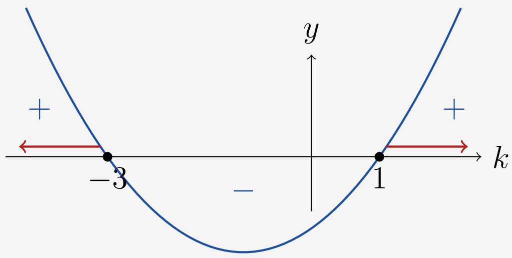
$$k^2 - 6k + 9 - 12 + 8k > 0$$

$$k^2 + 2k - 3 > 0 \qquad \checkmark$$

Part (b):

Solve $k^2 + 2k - 3 = 0$: $(k+3)(k-1) = 0$, so $k = -3$ or $k = 1$.

Since the coefficient of k^2 is positive, the parabola opens upward and the quadratic is positive outside the roots.



$k < -3 \quad \text{or} \quad k > 1$

Question 2

$f(x) = x^2 + (k+3)x + k$. (a) Find the discriminant. (b) Express in the form $(k+a)^2 + b$. (c) Show $f(x) = 0$ always has real roots.

Worked Solution

Part (a):

$$\Delta = (k+3)^2 - 4k = k^2 + 6k + 9 - 4k = k^2 + 2k + 9$$

$\Delta = k^2 + 2k + 9$

Part (b): Complete the square in k:

$$k^2 + 2k + 9 = (k+1)^2 - 1 + 9 = (k+1)^2 + 8$$

So $a = 1$, $b = 8$.

Part (c):

Since $(k+1)^2 \geq 0$ for all k, we have $\Delta = (k+1)^2 + 8 \geq 8 > 0$ for all values of k.

Therefore $f(x) = 0$ always has real roots. ✓

Question 3

$kx^2 + 4x + (5-k) = 0$ has two different real solutions. (a) Show $k^2 - 5k + 4 > 0$. (b) Find the set of possible values of k.

Worked Solution

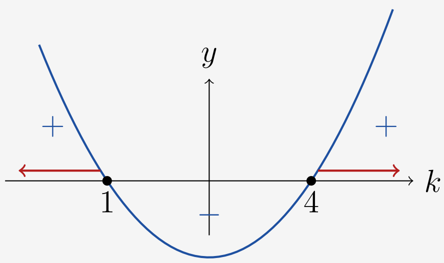
Part (a):

For two different real solutions, $b^2 - 4ac > 0$. Here $a = k$, $b = 4$, $c = 5-k$. Also $k \neq 0$.

$$16 - 4k(5-k) > 0$$
$$16 - 20k + 4k^2 > 0$$
$$4k^2 - 20k + 16 > 0$$
$$k^2 - 5k + 4 > 0 \quad \checkmark$$

Part (b):

Solve $k^2 - 5k + 4 = 0$: $(k-1)(k-4) = 0$, so $k = 1$ or $k = 4$.



We also need $k \neq 0$ (otherwise not a quadratic); since neither critical value is 0, this is automatically satisfied in each region.

$k < 1 \quad \text{or} \quad k > 4$

Question 4

$\mathrm{f}(x) = x^2 + 4kx + (3 + 11k)$. (a) Express in the form $(x + p)^2 + q$. (b) No real roots: find possible values of k. (c) Sketch $y = \mathrm{f}(x)$ for $k = 1$.

Worked Solution

Part (a):

$$f(x) = (x + 2k)^2 - 4k^2 + 3 + 11k$$

$p = 2k, \quad q = -4k^2 + 11k + 3$

Part (b):

For no real roots, $\Delta < 0$, i.e. $q < 0$ (using completed square) or equivalently $b^2 - 4ac < 0$:

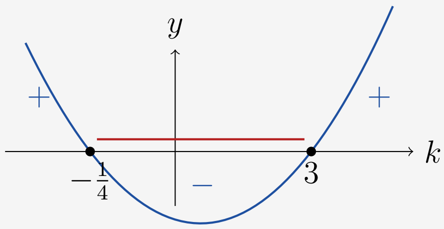
$$(4k)^2 - 4(3 + 11k) < 0$$

$$16k^2 - 12 - 44k < 0$$

$$4k^2 - 11k - 3 < 0$$

$$(4k + 1)(k - 3) < 0$$

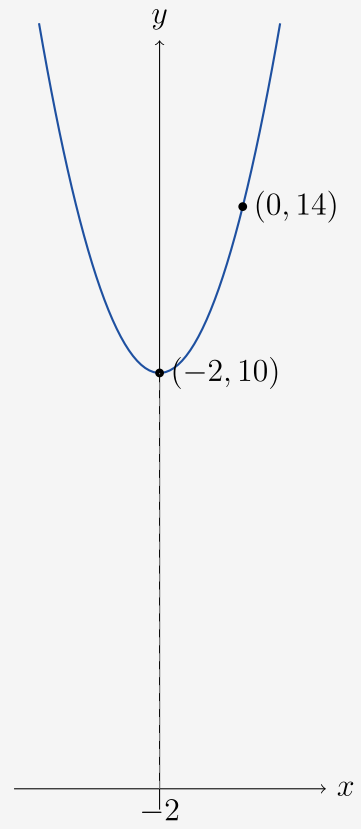
Critical values: $k = -\frac{1}{4}$ or $k = 3$.



$-\dfrac{1}{4} < k < 3$

Part (c): When $k = 1$: $\mathrm{f}(x) = x^2 + 4x + 14 = (x + 2)^2 + 10$.

No real roots (confirmed), minimum at $(-2, 10)$, y-intercept at $(0, 14)$.


y
(0, 14)
(−2, 10)
x
−2

Question 5

$x^2 + 3px + p = 0$ has equal roots ($p \neq 0$). Find p.

Worked Solution

Equal roots $\Rightarrow \Delta = 0$:

$$(3p)^2 - 4(1)(p) = 0$$

$$9p^2 - 4p = 0$$

$$p(9p - 4) = 0$$

Since $p \neq 0$: $9p = 4$.

$$p = \frac{4}{9}$$

Question 6

$(k+3)x^2 + 6x + k = 5$ has two distinct real solutions. (a) Show $k^2 - 2k - 24 < 0$. (b) Find the set of possible values of k.

Worked Solution

Part (a):

Rewrite: $(k+3)x^2 + 6x + (k-5) = 0$. Here $a = k+3$, $b = 6$, $c = k-5$.

For two distinct real solutions, $\Delta > 0$:

$$36 - 4(k+3)(k-5) > 0$$

$$36 - 4(k^2 - 2k - 15) > 0$$

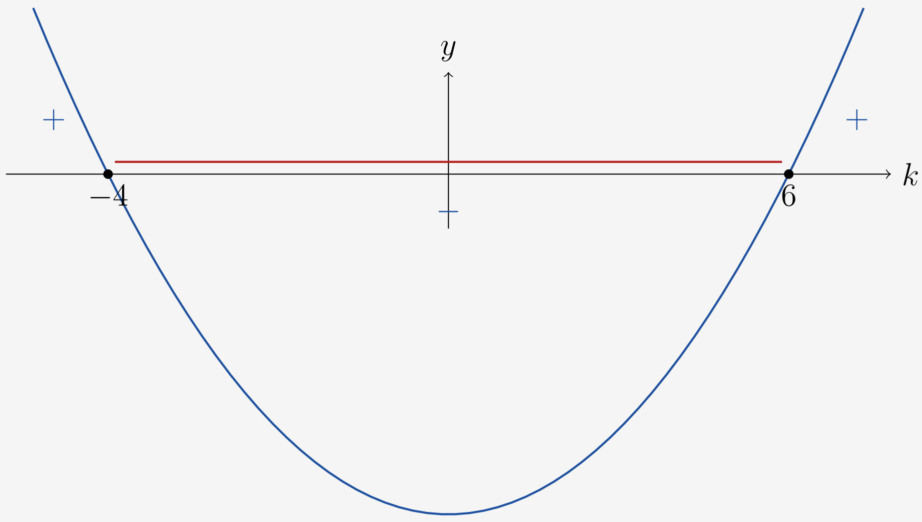
$$36 - 4k^2 + 8k + 60 > 0$$

$$-4k^2 + 8k + 96 > 0$$

$$k^2 - 2k - 24 < 0 \qquad \checkmark$$

Part (b):

Solve $k^2 - 2k - 24 = 0$: $(k-6)(k+4) = 0$, so $k = 6$ or $k = -4$.



$-4 < k < 6$

Question 7

Simultaneous equations: $2x+y=1$, $x^2-4ky+5k=0$, $k \neq 0$. (a) Show $x^2+8kx+k=0$. (b) Equal roots: find k. (c) Find the solution.

Worked Solution

Part (a):

From the linear: $y = 1 - 2x$. Substitute:

$$x^2 - 4k(1-2x) + 5k = 0$$

$$x^2 - 4k + 8kx + 5k = 0$$

$$x^2 + 8kx + k = 0 \qquad \checkmark$$

Part (b):

Equal roots $\Rightarrow \Delta = 0$:

$$(8k)^2 - 4k = 0$$

$$64k^2 - 4k = 0$$

$$4k(16k-1) = 0$$

Since $k \neq 0$: $k = \dfrac{1}{16}$.

$$k = \frac{1}{16}$$

Part (c):

With $k = \frac{1}{16}$: $x^2 + \frac{1}{2}x + \frac{1}{16} = 0$, i.e. $(x+\frac{1}{4})^2 = 0$, so $x = -\frac{1}{4}$.

Then $y = 1 - 2(-\frac{1}{4}) = 1 + \frac{1}{2} = \frac{3}{2}$.

$$x = -\frac{1}{4}, \quad y = \frac{3}{2}$$

Question 8

$2qx^2 + qx - 1 = 0$ has no real roots. (a) Show $q^2 + 8q < 0$. (b) Find the set of possible values of q.

Worked Solution

Part (a):

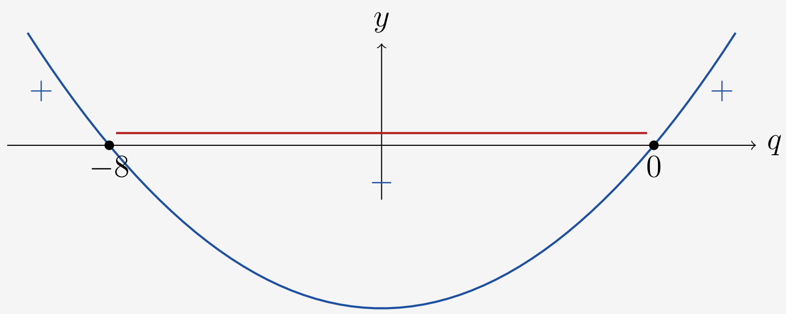
No real roots $\Rightarrow \Delta < 0$. Here $a = 2q$, $b = q$, $c = -1$:

$$q^2 - 4(2q)(-1) < 0$$

$$q^2 + 8q < 0 \qquad \checkmark$$

Part (b):

Solve $q^2 + 8q = 0$: $q(q+8) = 0$, so $q = 0$ or $q = -8$.



We also need $2q \neq 0$, i.e. $q \neq 0$, which is already excluded from the solution.

$-8 < q < 0$