Question 1 (Specimen Paper, Q5)

Worked Solution

(i) $y = x^2 - 3x + 2 = 3x - 7 \Rightarrow x^2 - 6x + 9 = 0 \Rightarrow (x-3)^2 = 0$, so $x = 3$, $y = 2$.

$x = 3,\ y = 2$

(ii) Only one solution $(x = 3, y = 2)$, so the line is tangent to the curve at $(3, 2)$.

(iii) Gradient of tangent $y = 3x - 7$ is 3, so gradient of normal $= -\frac{1}{3}$.

Normal at $(3, 2)$: $y - 2 = -\frac{1}{3}(x - 3) \Rightarrow 3y - 6 = -(x - 3) \Rightarrow x + 3y - 9 = 0$

$x + 3y - 9 = 0$

Question 2 (Jun 2005, Q10)

Worked Solution

$y = \frac{1}{3}x^3 - 9x$

(i) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = x^2 - 9$

(ii) $x = 3$: $y = -18$; $x = -3$: $y = 18$. (See Stationary Points Q1 for details.)

$(3, -18)$ and $(-3, 18)$.

(iii) At $x = 3$: $\frac{d^2y}{dx^2} = 6 > 0$ (min); at $x = -3$: $-6 < 0$ (max).

(iv) Gradient of $24x + 3y + 2 = 0$ is -8. $x^2 - 9 = -8 \Rightarrow x = \pm 1$.

Check both on curve and line: $x = 1$, $y = -\frac{26}{3}$ works.

$p = 1$, $q = -\dfrac{26}{3}$

Question 3 (Jan 2009, Q10)

Worked Solution

$y = x^2 + x$

(i) $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + 1$. At $x = 2$: gradient $= 5$.

Gradient $= 5$

(ii) Normal gradient $= -\frac{1}{5}$. At $x = 2$: $y = 6$.

$y - 6 = -\frac{1}{5}(x - 2) \Rightarrow 5y - 30 = -(x - 2) \Rightarrow x + 5y - 32 = 0$

$x + 5y - 32 = 0$

(iii) $y = kx - 4$ is tangent to $y = x^2 + x$: $x^2 + x = kx - 4 \Rightarrow x^2 + (1 - k)x + 4 = 0$.

For tangency, discriminant $= 0$: $(1 - k)^2 - 16 = 0 \Rightarrow (1 - k)^2 = 16 \Rightarrow 1 - k = \pm 4$.

$k = -3$ or $k = 5$

Question 4 (Jan 2010, Q3)

Worked Solution

$y = x^3 - 4x^2 + 7$ at $(2, -1)$.

$\frac{dy}{dx} = 3x^2 - 8x$. At $x = 2$: $12 - 16 = -4$.

Normal gradient $= \frac{1}{4}$.

$y + 1 = \frac{1}{4}(x - 2) \Rightarrow 4y + 4 = x - 2 \Rightarrow x - 4y - 6 = 0$

$x - 4y - 6 = 0$

Question 5 (Jan 2011, Q8)

Worked Solution

$y = 7 + 6x - x^2$

(i) At $x = 5$: $y = 7 + 30 - 25 = 12$. $\frac{dy}{dx} = 6 - 2x$. At $x = 5$: -4.

Tangent: $y - 12 = -4(x - 5) \Rightarrow y = -4x + 32 \Rightarrow 4x + y - 32 = 0$

$4x + y - 32 = 0$

(ii) Tangent meets x-axis: $4x - 32 = 0 \Rightarrow x = 8$, so $Q = (8, 0)$.

Midpoint of PQ: $\left(\frac{5+8}{2}, \frac{12+0}{2}\right)$

$\left(\frac{13}{2}, 6\right)$

(iii) Line of symmetry: $\frac{dy}{dx} = 0 \Rightarrow x = 3$.

$x = 3$

(iv) $7 + 6x - x^2$ increasing when $6 - 2x > 0$, i.e. $x < 3$.

$x < 3$

Question 6 (Jun 2011, Q10)

Worked Solution

$y = (2x-1)(x+3)(x-1)$

(i) Roots at $x = \frac{1}{2}, -3, 1$; y-intercept: $(2(-1))(3)(-1) = 6...$ at $x = 0$: $y = (-1)(3)(-1) = 3$.

Positive cubic through $(-3,0)$, $(0,3)$, $(\frac{1}{2},0)$, $(1,0)$.

(ii) Expand: $y = (2x-1)(x^2+2x-3) = 2x^3+4x^2-6x-x^2-2x+3 = 2x^3+3x^2-8x+3$.

$\frac{dy}{dx} = 6x^2+6x-8$. At $P(1,0)$: $6+6-8 = 4$.

Gradient at $P = 4$ ✓

(iii) Line l through $(-2, y(-2))$ with gradient 4: $y(-2) = (5)(1)(3) = 15$.

$y - 15 = 4(x+2)$

$y = 4x + 23$

(iv) Is l tangent at $x = -2$? Gradient at $x = -2$: $6(4)+6(-2)-8 = 24-12-8 = 4$. Yes, gradient matches. Check if l meets curve with repeated root at $x = -2$: substituting shows $(x+2)^2$ is a factor, so l is tangent to the curve at $x = -2$.

Yes, l is tangent to the curve at $x = -2$.

Question 7 (Jun 2012, Q6)

Worked Solution

$y = \dfrac{6}{x^2} - 5 = 6x^{-2} - 5$ at $x = 2$.

$y(2) = \frac{6}{4} - 5 = -\frac{7}{2}$. $\dfrac{dy}{dx} = -12x^{-3}$. At $x = 2$: $-\frac{12}{8} = -\frac{3}{2}$.

Normal gradient $= \frac{2}{3}$. $y + \frac{7}{2} = \frac{2}{3}(x-2) \Rightarrow 3y + \frac{21}{2} = 2x - 4 \Rightarrow 6y + 21 = 4x - 8$

$4x - 6y - 29 = 0$

Question 8 (Jun 2014, Q10)

Worked Solution

$y = (x+2)^2(2x-3)$. Roots at $x = -2$ (double) and $x = \frac{3}{2}$.

(i) Positive cubic: double root (touches) at -2, single root at $\frac{3}{2}$, y-intercept $(0, -12)$.

(ii) Expand: $y = (x^2+4x+4)(2x-3) = 2x^3 + 8x^2 + 8x - 3x^2 - 12x - 12 = 2x^3 + 5x^2 - 4x - 12$.

$\dfrac{dy}{dx} = 6x^2 + 10x - 4$. At $x = -1$: $6 - 10 - 4 = -8$.

$y(-1) = (1)(-5) = -5$.

Tangent: $y + 5 = -8(x+1) \Rightarrow 8x + y + 13 = 0$

$8x + y + 13 = 0$

Question 9 (Jun 2015, Q7)

Worked Solution

(a) $\mathrm{f}(x) = (x^2+3)(5-x) = 5x^2 + 15 - x^3 - 3x$

$\mathrm{f}'(x) = 10x - 3x^2 - 3$

$\mathrm{f}'(x) = -3x^2 + 10x - 3$

(b) $y = x^{-1/3}$. $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{3}x^{-4/3}$.

At $x = -8$: $(-8)^{-4/3} = \dfrac{1}{((-8)^{1/3})^4} = \dfrac{1}{(-2)^4} = \dfrac{1}{16}$.

Gradient $= -\dfrac{1}{3} \times \dfrac{1}{16}$

$-\dfrac{1}{48}$

Question 10 (Jun 2017, Q11)

Worked Solution

$y = \dfrac{k}{x^2} = kx^{-2}$. Normal at $x = -3$ parallel to $\frac{1}{2}y = 2 + 3x$ (gradient 6).

(i) Normal gradient $= 6 \Rightarrow$ tangent gradient $= -\frac{1}{6}$.

$\dfrac{dy}{dx} = -2kx^{-3}$. At $x = -3$: $-2k(-3)^{-3} = \dfrac{2k}{27} = -\frac{1}{6} \Rightarrow k = -\dfrac{27}{12}$

$k = -\dfrac{9}{4}$

(ii) At $x = -3$: $y = -\dfrac{9}{4} \times \dfrac{1}{9} = -\dfrac{1}{4}$.

Normal: $y + \frac{1}{4} = 6(x+3) \Rightarrow 4y + 1 = 24(x+3) = 24x + 72 \Rightarrow 24x - 4y + 71 = 0$

$24x - 4y + 71 = 0$

End of Worked Solutions