Question 1

Worked Solution

$y = x^3 - 2x^2 - x + 9$, $x > 0$. Point $P(2, 7)$.

(a) $y(2) = 8 - 8 - 2 + 9 = 7$. ✓

(b) $\frac{dy}{dx} = 3x^2 - 4x - 1$. At $x = 2$: $12 - 8 - 1 = 3$.

Tangent: $y - 7 = 3(x - 2)$

$y = 3x + 1$

(c) Normal gradient at $P = -\frac{1}{3}$. At Q: gradient $= -\frac{1}{3}$, so $3x^2 - 4x - 1 = -\frac{1}{3}$, i.e. $9x^2 - 12x - 3 = -1$, giving $9x^2 - 12x - 2 = 0$.

$x = \frac{12 \pm \sqrt{144 + 72}}{18} = \frac{12 \pm \sqrt{216}}{18} = \frac{12 \pm 6\sqrt{6}}{18} = \frac{2 \pm \frac{\sqrt{6}}{1}}{3}$.

Actually: $\sqrt{216} = 6\sqrt{6}$, so $x = \frac{12 \pm 6\sqrt{6}}{18} = \frac{2 \pm \frac{1}{3}\sqrt{6} \cdot 3}{3}$. Let me simplify: $x = \frac{12 \pm 6\sqrt{6}}{18} = \frac{2 \pm \sqrt{6}/3 \cdot 3}{3}$...

$x = \frac{12 \pm 6\sqrt{6}}{18} = \frac{6(2 \pm \sqrt{6})}{18} = \frac{2 \pm \sqrt{6}}{3}$. But also we can write $x = \frac{1}{3}(2 \pm \sqrt{6})$. Since $Q \neq P$ (where $x = 2$), take the $+$ root:

x-coordinate of $Q = \frac{1}{3}(2 + \sqrt{6})$ ✓

Question 2

Worked Solution

$y = kx^3 - x^2 + x - 5$

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3kx^2 - 2x + 1$

(b) Gradient of $2y - 7x + 1 = 0$ is $\dfrac{7}{2}$. At $x = -\frac{1}{2}$:

$3k\left(\frac{1}{4}\right) - 2\left(-\frac{1}{2}\right) + 1 = \dfrac{3k}{4} + 2 = \dfrac{7}{2} \Rightarrow \dfrac{3k}{4} = \dfrac{3}{2} \Rightarrow k = 2.$

$k = 2$

(c) $x = -\frac{1}{2}$: $y = 2(-\frac{1}{8}) - (\frac{1}{4}) + (-\frac{1}{2}) - 5 = -\frac{1}{4} - \frac{1}{4} - \frac{1}{2} - 5 = -6.$

y-coordinate of $A = -6$

Question 3

Worked Solution

$y = x^2(x-6) + \dfrac{4}{x} = x^3 - 6x^2 + 4x^{-1}$

(a) At $x = 1$: $y = 1 - 6 - 4 = -1 \cdot (-1)\ldots$ $y = 1 - 6 + 4 = -1$. At $x = 2$: $y = 8 - 24 + 2 = -14$.

$PQ = \sqrt{(2-1)^2 + (-14-(-1))^2} = \sqrt{1+169} = \sqrt{170}$ ✓

(b) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 12x - 4x^{-2}$.

At $x = 1$: $3 - 12 - 4 = -13$. At $x = 2$: $12 - 24 - 1 = -13$. Same gradient $\Rightarrow$ parallel. ✓

(c) Normal gradient $= \dfrac{1}{13}$. Through $P(1, -1)$:

$y + 1 = \dfrac{1}{13}(x-1) \Rightarrow 13y + 13 = x - 1$

$x - 13y - 14 = 0$

Question 4

Worked Solution

$y = 2x^3 + kx^2 + 5x + 6$

(a) $\dfrac{dy}{dx} = 6x^2 + 2kx + 5$

(b) Gradient of $2y - 17x - 1 = 0$ is $\dfrac{17}{2}$. At $x = -2$: $6(4) + 2k(-2) + 5 = 24 - 4k + 5 = \dfrac{17}{2}$.

$29 - 4k = \dfrac{17}{2} \Rightarrow 4k = 29 - \dfrac{17}{2} = \dfrac{41}{2} \Rightarrow k = \dfrac{41}{8}$.

$k = \dfrac{41}{8}$

(c) $y(-2) = 2(-8) + \dfrac{41}{8}(4) + 5(-2) + 6 = -16 + \dfrac{41}{2} - 10 + 6 = -20 + \dfrac{41}{2} = \dfrac{1}{2}$.

y-coordinate of $P = \dfrac{1}{2}$

(d) Tangent at $P(-2, \frac{1}{2})$, gradient $\dfrac{17}{2}$:

$y - \frac{1}{2} = \frac{17}{2}(x + 2) \Rightarrow 2y - 1 = 17(x + 2) \Rightarrow 17x - 2y + 35 = 0$

$-17x + 2y - 35 = 0$ (or $17x - 2y + 35 = 0$)

Question 5

Worked Solution

$y = \frac{1}{2}x^3 - 9x^{3/2} + \dfrac{8}{x} + 30 = \frac{1}{2}x^3 - 9x^{3/2} + 8x^{-1} + 30$

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{2}x^2 - \frac{27}{2}x^{1/2} - 8x^{-2}$

(b) At $x = 4$: $\frac{1}{2}(64) - 9(8) + 2 + 30 = 32 - 72 + 2 + 30 = -8$. ✓

(c) At $x = 4$: $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{2}(16) - \frac{27}{2}(2) - \frac{8}{16} = 24 - 27 - \frac{1}{2} = -\frac{7}{2}$.

Normal gradient $= \dfrac{2}{7}$. Through $(4, -8)$:

$y + 8 = \dfrac{2}{7}(x - 4) \Rightarrow 7y + 56 = 2x - 8 \Rightarrow 7y - 2x + 64 = 0$

$7y - 2x + 64 = 0$

Question 6

Worked Solution

$y = 2x - 8\sqrt{x} + 5 = 2x - 8x^{1/2} + 5$

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2 - 4x^{-1/2} = 2 - \dfrac{4}{\sqrt{x}}$

(b) At $x = \frac{1}{4}$: $y = \frac{1}{2} - 8(\frac{1}{2}) + 5 = \frac{1}{2} - 4 + 5 = \frac{3}{2}$. Gradient $= 2 - \dfrac{4}{1/2} = 2 - 8 = -6$.

$y - \frac{3}{2} = -6(x - \frac{1}{4})$

$y = -6x + 3$

(c) Tangent at Q parallel to $2x - 3y + 18 = 0$ (gradient $\frac{2}{3}$): $2 - \dfrac{4}{\sqrt{x}} = \dfrac{2}{3}$.

$\dfrac{4}{\sqrt{x}} = \dfrac{4}{3} \Rightarrow \sqrt{x} = 3 \Rightarrow x = 9$.

$y = 18 - 24 + 5 = -1$.

$Q = (9, -1)$

Question 7

Worked Solution

$\mathrm{f}'(x) = 30 + \dfrac{6 - 5x^2}{\sqrt{x}} = 30 + 6x^{-1/2} - 5x^{3/2}$

(a) At $x = 4$: $\mathrm{f}'(4) = 30 + \dfrac{6 - 80}{2} = 30 + \dfrac{-74}{2} = 30 - 37 = -7$.

Tangent at $(4, -8)$: $y + 8 = -7(x - 4)$

$y = -7x + 20$

(b) $\mathrm{f}(x) = \int \mathrm{f}'(x)\,\mathrm{d}x = 30x + 12x^{1/2} - 2x^{5/2} + C$.

At $(4, -8)$: $-8 = 120 + 24 - 64 + C \Rightarrow C = -88$.

$\mathrm{f}(x) = 30x + 12x^{1/2} - 2x^{5/2} - 88$

Question 8

Worked Solution

$y = 9 - 4x - \frac{8}{x} = 9 - 4x - 8x^{-1}$

(a) At $x = 2$: $y = 9 - 8 - 4 = -3$. $\frac{\mathrm{d}y}{\mathrm{d}x} = -4 + 8x^{-2}$. At $x = 2$: $-4 + 2 = -2$.

Tangent: $y + 3 = -2(x - 2)$

$y = 1 - 2x$ ✓

(b) Normal gradient $= \frac{1}{2}$. $y + 3 = \frac{1}{2}(x - 2)$

$y = \frac{1}{2}x - 4$

(c) A: tangent meets x-axis: $0 = 1 - 2x \Rightarrow x = \frac{1}{2}$, so $A = (\frac{1}{2}, 0)$.

B: normal meets x-axis: $0 = \frac{1}{2}x - 4 \Rightarrow x = 8$, so $B = (8, 0)$.

$AB = 8 - \frac{1}{2} = \frac{15}{2}$. Height of triangle (at P): $|y_P| = 3$.

Area $= \frac{1}{2} \times \frac{15}{2} \times 3$

Area $= \frac{45}{4} = 11.25$

Question 9

Worked Solution

$y = 20 - 4x - \dfrac{18}{x}$

(a) At $x = 2$: $y = 20 - 8 - 9 = 3$. $\dfrac{dy}{dx} = -4 + 18x^{-2}$. At $x = 2$: $-4 + \frac{18}{4} = -4 + \frac{9}{2} = \frac{1}{2}$.

Normal gradient $= -2$. Check $y = -2x + 7$ at $(2, 3)$: $-4 + 7 = 3$. ✓

Normal: $y = -2x + 7$ ✓

(b) Substitute normal into curve: $20 - 4x - \dfrac{18}{x} = -2x + 7$.

$13 - 2x = \dfrac{18}{x} \Rightarrow x(13 - 2x) = 18 \Rightarrow 2x^2 - 13x + 18 = 0 \Rightarrow (2x - 9)(x - 2) = 0$.

$x = \frac{9}{2}$: $y = -9 + 7 = -2$.

$B = \left(\dfrac{9}{2}, -2\right)$

Question 10

Worked Solution

$y = \frac{3}{x} + 4$

(a) Crosses x-axis when $y = 0$: $\frac{3}{x} = -4 \Rightarrow x = -\frac{3}{4}$.

$\left(-\frac{3}{4}, 0\right)$

(b) Asymptotes: $x = 0$ (vertical) and $y = 4$ (horizontal).

$x = 0$ and $y = 4$

(c) $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{3}{x^2}$. At $x = -3$: gradient $= -\frac{3}{9} = -\frac{1}{3}$. Normal gradient $= 3$.

Normal at $(-3, 3)$: $y - 3 = 3(x + 3)$

$y = 3x + 12$

(d) A: $y = 0$: $x = -4$, so $A = (-4, 0)$. B: $x = 0$: $y = 12$, so $B = (0, 12)$.

$AB = \sqrt{16 + 144} = \sqrt{160}$

$AB = \sqrt{160} = 4\sqrt{10}$

Question 11

Worked Solution

$y = \frac{1}{2}x + \dfrac{27}{x} - 12, \quad A = \left(3, -\frac{3}{2}\right)$

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2} - \dfrac{27}{x^2}$. At $x = 3$: $\frac{1}{2} - 3 = -\frac{5}{2}$.

Normal gradient $= \dfrac{2}{5}$.

$y + \frac{3}{2} = \dfrac{2}{5}(x-3) \Rightarrow 5y + \dfrac{15}{2} = 2x - 6 \Rightarrow 10y + 15 = 4x - 12$

$10y = 4x - 27$ ✓

(b) Substitute $\frac{1}{2}x + \dfrac{27}{x} - 12 = \dfrac{4x-27}{10}$:

$10\left(\frac{1}{2}x + \dfrac{27}{x} - 12\right) = 4x - 27 \Rightarrow 5x + \dfrac{270}{x} - 120 = 4x - 27$

$x + \dfrac{270}{x} - 93 = 0 \Rightarrow x^2 - 93x + 270 = 0 \Rightarrow (x-90)(x-3) = 0.$

$x = 90$: $y = \dfrac{360-27}{10} = \dfrac{333}{10} = 33.3.$

$B = \left(90, \dfrac{333}{10}\right)$

Question 12

Worked Solution

$y = \frac{12}{x} + 5 = 12x^{-1} + 5$

(a) $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{12}{x^2}$. At $A(-2, -1)$: gradient $= -\frac{12}{4} = -3$. Normal gradient $= \frac{1}{3}$.

$y + 1 = \frac{1}{3}(x + 2) \Rightarrow 3y + 3 = x + 2$

$3y - x + 1 = 0$

(b) Normal at B and C parallel to $4y = 3x + 5$, so gradient $= \frac{3}{4}$. Normal gradient $= \frac{3}{4}$ means tangent gradient $= -\frac{4}{3}$.

$-\frac{12}{x^2} = -\frac{4}{3} \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$.

$x = 3$: $y = 4 + 5 = 9$. $\quad x = -3$: $y = -4 + 5 = 1$.

B has positive x-coordinate.

$B = (3, 9)$ and $C = (-3, 1)$.

End of Worked Solutions