Question 1

Worked Solution

$y = 2x + 3 + \frac{8}{x^2} = 2x + 3 + 8x^{-2}, \quad x > 0$

$\frac{dy}{dx} = 2 - 16x^{-3} = 2 - \frac{16}{x^3}$

Set to zero: $2 = \frac{16}{x^3} \Rightarrow x^3 = 8 \Rightarrow x = 2$.

$y = 4 + 3 + \frac{8}{4} = 9$.

Stationary point at $(2, 9)$.

Question 2

Worked Solution

$y = 6 - 3x - 4x^{-3}$

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -3 + 12x^{-4} = -3 + \dfrac{12}{x^4}$

Set to zero: $\dfrac{12}{x^4} = 3 \Rightarrow x^4 = 4 \Rightarrow x = \pm\sqrt{2}$.

At $x = \sqrt{2}$: $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ ✓

(b) Other turning point: $x = -\sqrt{2}$.

$x = -\sqrt{2}$

(c) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\dfrac{48}{x^5}$

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(d) At P $(x = \sqrt{2})$: $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\dfrac{48}{(\sqrt{2})^5} = -\dfrac{48}{4\sqrt{2}} < 0 \Rightarrow$ maximum.

At Q $(x = -\sqrt{2})$: $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\dfrac{48}{(-\sqrt{2})^5} = \dfrac{48}{4\sqrt{2}} > 0 \Rightarrow$ minimum.

P is a maximum; Q is a minimum.

Question 3

Worked Solution

$y = x^2 - 32\sqrt{x} + 20 = x^2 - 32x^{1/2} + 20, \quad x > 0$

(a) $\dfrac{dy}{dx} = 2x - 16x^{-1/2}$

Set to zero: $2x = \dfrac{16}{x^{1/2}} \Rightarrow 2x^{3/2} = 16 \Rightarrow x^{3/2} = 8 \Rightarrow x = 4.$

$y = 16 - 32(2) + 20 = 16 - 64 + 20 = -28.$

Stationary point $P = (4, -28)$.

(b) $\dfrac{d^2y}{dx^2} = 2 + 8x^{-3/2}$. At $x = 4$: $= 2 + \dfrac{8}{8} = 3 > 0.$

P is a minimum.

Question 4

Worked Solution

$y = 12\sqrt{x} - x^{3/2} - 10 = 12x^{1/2} - x^{3/2} - 10, \quad x > 0$

(a) $\dfrac{dy}{dx} = 6x^{-1/2} - \dfrac{3}{2}x^{1/2} = \dfrac{6}{\sqrt{x}} - \dfrac{3\sqrt{x}}{2}$

Set to zero: multiply through by $\sqrt{x}$: $6 - \dfrac{3x}{2} = 0 \Rightarrow x = 4$.

$y = 12(2) - 4^{3/2} - 10 = 24 - 8 - 10 = 6$.

Turning point at $(4, 6)$.

(b) $\dfrac{d^2y}{dx^2} = -3x^{-3/2} - \dfrac{3}{4}x^{-1/2}$

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(c) At $x = 4$: $\dfrac{d^2y}{dx^2} = -\dfrac{3}{8} - \dfrac{3}{8} = -\dfrac{3}{4} < 0$, so it is a maximum.

Maximum point.

Question 5

Worked Solution

$f(x) = ax^3 + 15x^2 - 39x + b$

(a)(i) $f'(x) = 3ax^2 + 30x - 39$. At $(2, 10)$ with gradient -3:

$$3a(4) + 60 - 39 = -3 \Rightarrow 12a + 21 = -3 \Rightarrow 12a = -24$$

$a = -2$ ✓

(a)(ii) $f(2) = 10$: $-2(8) + 15(4) - 39(2) + b = 10 \Rightarrow -16 + 60 - 78 + b = 10 \Rightarrow b = 44$.

$b = 44$

(b) $f'(x) = -6x^2 + 30x - 39$. Discriminant: $30^2 - 4(-6)(-39) = 900 - 936 = -36 < 0$.

Since $-6x^2 + 30x - 39$ has no real roots and leading coefficient is negative, $f'(x) < 0$ for all x.

$f'(x) \neq 0$ for any real x, so C has no stationary points.

(c) $f(x) = -2x^3 + 15x^2 - 39x + 44$. Try $x = 4$: $-128 + 240 - 156 + 44 = 0$. ✓

$f(x) = (x - 4)(-2x^2 + 7x - 11)$

(d) $y = f(0.2x) = (0.2x - 4)(-2(0.04x^2) + 7(0.2x) - 11)$.

y-intercept: $x = 0 \Rightarrow y = f(0) = 44$, so $(0, 44)$.

x-intercept: $0.2x - 4 = 0 \Rightarrow x = 20$, so $(20, 0)$.

$(0, 44)$ and $(20, 0)$.

Question 6

Worked Solution

$g(x) = ax^3 + bx^2 + ax$ (coefficient of x^3 equals coefficient of x; passes through origin).

(a) Stationary point at $(2, 9)$: $g(2) = 9$ and $g'(2) = 0$.

$g'(x) = 3ax^2 + 2bx + a$.

From $g'(2) = 0$: $12a + 4b + a = 0 \Rightarrow 13a + 4b = 0$. (1)

From $g(2) = 9$: $8a + 4b + 2a = 9 \Rightarrow 10a + 4b = 9$. (2)

(1)−(2): $3a = -9 \Rightarrow a = -3$, then $4b = 9 - 10(-3) = 39 \Rightarrow b = \dfrac{39}{4}$.

$$g(x) = -3x^3 + \frac{39}{4}x^2 - 3x$$

(b) $g''(x) = -18x + \dfrac{39}{2}$. At $x = 2$: $g''(2) = -36 + \dfrac{39}{2} = -\dfrac{33}{2} < 0$.

$g''(2) < 0$, so $(2, 9)$ is a maximum.

End of Worked Solutions