Question 1

Worked Solution

We have

$$\frac{9^{x-1}}{3^{y+2}} = 81.$$

Step 1: Write everything as a power of 3.

Note $9 = 3^2$, so $9^{x-1} = 3^{2(x-1)} = 3^{2x-2}$, and $81 = 3^4$. The equation becomes:

$$\frac{3^{2x-2}}{3^{y+2}} = 3^4.$$

Step 2: Apply the subtraction index law.

$$3^{(2x-2)-(y+2)} = 3^4 \implies 3^{2x-y-4} = 3^4.$$

Step 3: Equate exponents.

$$2x - y - 4 = 4 \implies y = 2x - 8.$$

$$\boxed{y = 2x - 8}$$

Question 2

Worked Solution

Part (a): Solve $5^x = 7$.

Take logarithms of both sides:

$$x \log 5 = \log 7 \implies x = \frac{\log 7}{\log 5}.$$

$$x = \frac{\log 7}{\log 5} \approx 1.21$$

Part (b): Solve $5^{2x} - 12(5^x) + 35 = 0$.

Step 1: Substitution. Let $u = 5^x$, so $5^{2x} = u^2$:

$$u^2 - 12u + 35 = 0.$$

Step 2: Factorise.

$$(u-7)(u-5) = 0 \implies u = 7 \text{ or } u = 5.$$

Step 3: Solve for x.

If $5^x = 7$: from part (a), $x = \frac{\log 7}{\log 5} \approx 1.21$.

If $5^x = 5 = 5^1$: $x = 1$.

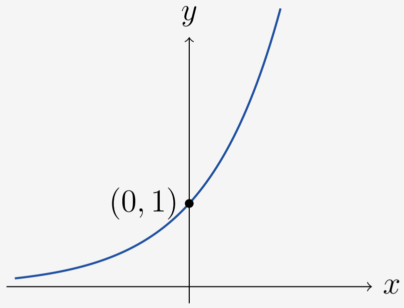
$$x \approx 1.21 \quad \text{or} \quad x = 1$$

Question 3

Worked Solution

Part (a): Sketch $y = 3^x$, $x \in \mathbb{R}$.

Key features: passes through $(0, 1)$; increasing for $x > 0$, approaching 0 (never touching the x-axis) for $x < 0$; no x-intercept.



The graph crosses the y-axis at $(0, 1)$; it does not cross the x-axis.

Part (b): Solve $3^{2x} - 9(3^x) + 18 = 0$.

Step 1: Substitution. Let $y = 3^x$, so $3^{2x} = y^2$:

$$y^2 - 9y + 18 = 0.$$

Step 2: Factorise.

$$(y - 6)(y - 3) = 0 \implies y = 6 \text{ or } y = 3.$$

Step 3: Solve for x.

If $3^x = 6$:

$$x \log 3 = \log 6 \implies x = \frac{\log 6}{\log 3} = \log_3 6 \approx 1.63.$$

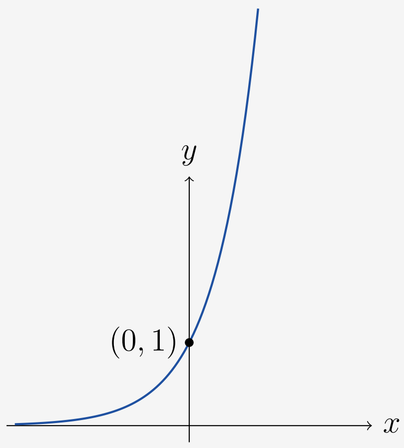
If $3^x = 3 = 3^1$: $x = 1$.

$$x = 1 \quad \text{or} \quad x = \log_3 6 \approx 1.63$$

Question 4

Worked Solution

Part (a): Sketch $y = 7^x$, $x \in \mathbb{R}$.



The graph crosses the y-axis at $(0, 1)$; it does not cross the x-axis.

Part (b): Solve $7^{2x} - 4(7^x) + 3 = 0$.

Step 1: Substitution. Let $y = 7^x$, so $7^{2x} = y^2$:

$$y^2 - 4y + 3 = 0.$$

Step 2: Factorise.

$$(y - 3)(y - 1) = 0 \implies y = 3 \text{ or } y = 1.$$

Step 3: Solve for x.

If $7^x = 3$:

$$x \log 7 = \log 3 \implies x = \frac{\log 3}{\log 7} = \log_7 3 \approx 0.565.$$

If $7^x = 1 = 7^0$: $x = 0$.

$$x \approx 0.565 \quad \text{or} \quad x = 0$$

Question 5

Worked Solution

$f(x) = -6x^3 - 7x^2 + 40x + 21$.

Part (a): Show $(x + 3)$ is a factor.

Use the factor theorem: compute $f(-3)$:

$$f(-3) = -6(-27) - 7(9) + 40(-3) + 21 = 162 - 63 - 120 + 21 = 0.$$

Since $f(-3) = 0$, $(x + 3)$ is a factor. □

Part (b): Factorise $f(x)$ completely.

Divide $f(x)$ by $(x + 3)$ to obtain:

$$f(x) = (x + 3)(-6x^2 + 11x + 7).$$

Now factorise $-6x^2 + 11x + 7$:

$$-6x^2 + 11x + 7 = (-3x + 7)(2x + 1) = -(3x - 7)(2x + 1).$$

Therefore:

$$f(x) = -(x + 3)(3x - 7)(2x + 1).$$

$$\boxed{f(x) = -(x + 3)(3x - 7)(2x + 1)}$$

Part (c): Solve $6(2^{3y}) + 7(2^{2y}) = 40(2^y) + 21$.

Rearrange:

$$6(2^{3y}) + 7(2^{2y}) - 40(2^y) - 21 = 0.$$

Let $x = 2^y$, so $2^{3y} = x^3$, $2^{2y} = x^2$. This gives $f(x) = -6x^3 - 7x^2 + 40x + 21 = 0$ (adjusting signs: multiply through by -1 to get $6x^3 + 7x^2 - 40x - 21 = 0$, which is $-f(x) = 0$, same solutions).

The roots of $f(x) = 0$ are $x = -3$, $x = \frac{7}{3}$, $x = -\frac{1}{2}$.

Since $2^y > 0$, we reject $x = -3$ and $x = -\frac{1}{2}$.

So $2^y = \frac{7}{3}$:

$$y \log 2 = \log\left(\frac{7}{3}\right) \implies y = \frac{\log(7/3)}{\log 2} = \log_2\left(\frac{7}{3}\right).$$

$$y \approx 1.22.$$

$$\boxed{y = \log_2\left(\frac{7}{3}\right) \approx 1.22}$$

Question 6

Worked Solution

$f(x) = 2x^3 - 5x^2 + ax + 18$, where $(x-3)$ is a factor.

Part (a): Show $a = -9$.

By the factor theorem, $f(3) = 0$:

$$2(27) - 5(9) + 3a + 18 = 0 \implies 54 - 45 + 3a + 18 = 0 \implies 3a = -27 \implies a = -9. \quad \square$$

Part (b): Factorise $f(x) = 2x^3 - 5x^2 - 9x + 18$ completely.

Divide by $(x-3)$:

$$f(x) = (x-3)(2x^2 + x - 6).$$

Factorise the quadratic:

$$2x^2 + x - 6 = (2x-3)(x+2).$$

Therefore:

$$f(x) = (x-3)(2x-3)(x+2).$$

$$\boxed{f(x) = (x-3)(2x-3)(x+2)}$$

Part (c): Solve $g(y) = 2(3^{3y}) - 5(3^{2y}) - 9(3^y) + 18 = 0$.

Let $x = 3^y$. Then $g(y) = f(x) = 0$, giving $x = 3$, $x = \frac{3}{2}$, $x = -2$.

Since $3^y > 0$, reject $x = -2$.

If $3^y = 3$: $y = 1$.

If $3^y = \frac{3}{2}$:

$$y \log 3 = \log\left(\tfrac{3}{2}\right) \implies y = \frac{\log(3/2)}{\log 3} = \log_3\left(\tfrac{3}{2}\right) \approx 0.37.$$

$$\boxed{y = 1 \quad \text{or} \quad y = \log_3\left(\tfrac{3}{2}\right) \approx 0.37}$$

End of Worked Solutions