Question 1

Worked Solution

Part (i) P: $x^2 + x - 2 = 0$, Q: $x = 1$

Factorise P: $x^2 + x - 2 = (x+2)(x-1) = 0$, so $x = 1$ or $x = -2$.

Q states $x = 1$, but P also allows $x = -2$. So Q is *sufficient* for P (if $x = 1$ then P holds), but P does *not* imply Q (P can hold with $x = -2$).

$$P \Leftarrow Q$$

Part (ii) P: $y^3 > 1$, Q: $y > 1$

If $y > 1$ then $y^3 > 1^3 = 1$, so $Q \Rightarrow P$.

If $y^3 > 1$ then $y > 1$ (since $f(y) = y^3$ is strictly increasing), so $P \Rightarrow Q$.

Both directions hold.

$$P \Leftrightarrow Q$$

Question 2

Worked Solution

Write down the converse:

The original statement is: *"n is an odd integer $\Rightarrow$ $2n$ is an even integer."*

The converse reverses the implication:

"$2n$ is an even integer $\Rightarrow$ n is an odd integer."

Show the converse is false:

A statement is false if we can find a counterexample — a case where the hypothesis is true but the conclusion is false.

Let $n = 2$ (an *even* integer). Then $2n = 4$, which is an even integer.

So the hypothesis "$2n$ is an even integer" is satisfied, but the conclusion "n is an odd integer" is **false**.

This is a counterexample, so the converse is false. □

Question 3

Worked Solution

Factorise $n^3 + 3n^2 + 2n$:

$$n^3 + 3n^2 + 2n = n(n^2 + 3n + 2) = n(n + 1)(n + 2)$$

Prove divisible by 6 for all positive integers n:

$n(n + 1)(n + 2)$ is the product of three *consecutive* integers.

Among any three consecutive integers:

- **At least one must be even** (consecutive integers alternate odd/even), so the product is divisible by 2.
- **Exactly one must be a multiple of 3** (every third integer is divisible by 3), so the product is divisible by 3.

Since the product is divisible by both 2 and 3, and $\gcd(2, 3) = 1$, it is divisible by $2 \times 3 = 6$.

Therefore $n^3 + 3n^2 + 2n$ is always divisible by 6 for any positive integer n. □

Question 4

Worked Solution

Part (i) "$n^3 + 1$ is an odd integer" ☐ "n is an even integer"

$\Rightarrow$ direction: If n is even, then n^3 is even (even × even × even), so $n^3 + 1$ is odd. ✓

$\Leftarrow$ direction: If $n^3 + 1$ is odd, then n^3 is even, which means n must be even (since if n were odd, n^3 would be odd). ✓

Both directions hold, so the correct symbol is:

$$n^3 + 1 \text{ is an odd integer} \iff n \text{ is an even integer}$$

Part (ii) "$(x-3)(x-2) > 0$" ☐ "$x > 3$"

$\Rightarrow$ direction (does $x > 3$ imply $(x-3)(x-2) > 0$?): If $x > 3$, both factors $(x-3)$ and $(x-2)$ are positive, so their product is positive. ✓

$\Leftarrow$ direction (does $(x-3)(x-2) > 0$ imply $x > 3$?): $(x-3)(x-2) > 0$ also holds when $x < 2$ (both factors negative, product positive). For example, $x = 0$: $(0-3)(0-2) = (-3)(-2) = 6 > 0$, but $x = 0 \not> 3$.

So $\Leftarrow$ is **false**. The correct symbol is:

$$(x-3)(x-2) > 0 \Longleftarrow x > 3 \qquad \text{i.e. } \Leftarrow$$

Question 5

Worked Solution

The three consecutive integers are $n-1$, n and $n+1$.

Part (i) — Sum always divisible by 3:

$$(n-1)+n+(n+1)=3n$$

$3n$ is always a multiple of 3 for any integer n. □

Part (ii) — Sum of squares never divisible by 3:

$$\begin{aligned}&(n-1)^2+n^2+(n+1)^2\\ &=(n^2-2n+1)+n^2+(n^2+2n+1)\\ &=3n^2+2\end{aligned}$$

Now consider divisibility by 3: $3n^2$ is always divisible by 3, so $3n^2+2$ always leaves a remainder of 2 when divided by 3.

Therefore $3n^2+2$ is **never** divisible by 3, regardless of the value of n. □

Question 6

Worked Solution

The three consecutive integers are n, $n+1$, $n+2$.

Part (i) — Expand $n^2+(n+1)^2+(n+2)^2$:

$$n^2+(n+1)^2+(n+2)^2$$

$$= n^2+(n^2+2n+1)+(n^2+4n+4)$$
$$= 3n^2+6n+5$$

$$3n^2+6n+5$$

Part (ii) — For what values of n is this even?

Write $3n^2+6n+5 = 3n^2+6n+4+1 = 3n(n+2)+5$.

Alternatively, note $6n$ is always even and 5 is odd, so the parity of $3n^2+6n+5$ depends only on $3n^2$.

$3n^2$ is even $\iff$ n^2 is even $\iff$ n is even.

When n is even: $3n^2+6n+5 = \text{even}+\text{even}+\text{odd} = \text{odd}$.

When n is odd: $3n^2+6n+5 = \text{odd}+\text{even}+\text{odd} = \text{even}$.

The sum is even when n is **odd**.

Question 7

Worked Solution

Positive integers a, b, c form a Pythagorean triple if $a^2 + b^2 = c^2$.

Part (i) — Show $2t$, $t^2 - 1$, $t^2 + 1$ form a Pythagorean triple for integer $t > 1$:

We need to show $(2t)^2 + (t^2 - 1)^2 = (t^2 + 1)^2$.

LHS:

$$(2t)^2 + (t^2 - 1)^2 = 4t^2 + t^4 - 2t^2 + 1 = t^4 + 2t^2 + 1$$

RHS:

$$(t^2 + 1)^2 = t^4 + 2t^2 + 1$$

LHS = RHS, so $2t$, $t^2 - 1$, $t^2 + 1$ satisfy $a^2 + b^2 = c^2$ and therefore form a Pythagorean triple. □

Part (ii) — Two smallest integers are 20 and 21; find third, and show not all triples fit this form:

Find the third integer:

$$c = \sqrt{20^2 + 21^2} = \sqrt{400 + 441} = \sqrt{841} = 29$$

The triple is $\{20, 21, 29\}$.

Show this cannot be expressed as $2t$, $t^2 - 1$, $t^2 + 1$:

If $2t = 20$ then $t = 10$, giving $t^2 - 1 = 99$ and $t^2 + 1 = 101$. These are not 21 and 29.

Alternatively, in the form $2t$, $t^2 - 1$, $t^2 + 1$, the two larger values differ by:

$$(t^2 + 1) - (t^2 - 1) = 2$$

But $29 - 21 = 8 \neq 2$, so the triple $(20, 21, 29)$ cannot be expressed in this form.

Therefore not all Pythagorean triples can be expressed as $2t$, $t^2 - 1$, $t^2 + 1$. □

Question 8

Worked Solution

Prove the statement is false:

The statement claims $n^2 + 3n + 1$ is prime for all integers $n \geq 1$.

To disprove it, we only need one counterexample.

Test values:

n	$n^2 + 3n + 1$	Prime?
1	5	Yes
2	11	Yes
3	19	Yes
4	29	Yes
5	41	Yes
6	$55 = 5 \times 11$	**No**

When $n = 6$: $n^2 + 3n + 1 = 36 + 18 + 1 = 55 = 5 \times 11$, which is **not prime**.

This counterexample disproves the statement. □

Question 9

Worked Solution

Part (i)(A) — Show $(x-y)(x^2+xy+y^2)=x^3-y^3$:

Expand the left-hand side:

$$\begin{aligned}&(x-y)(x^2+xy+y^2)\\&=x^3+x^2y+xy^2-x^2y-xy^2-y^3\\&=x^3-y^3\square\end{aligned}$$

Part (i)(B) — Show $\left(x+\frac{1}{2}y\right)^2+\frac{3}{4}y^2=x^2+xy+y^2$:

Expand the left-hand side:

$$\left(x+\tfrac{1}{2}y\right)^2+\tfrac{3}{4}y^2=x^2+xy+\tfrac{1}{4}y^2+\tfrac{3}{4}y^2=x^2+xy+y^2\square$$

Part (ii) — Prove: for all real x, y, if $x>y$ then $x^3>y^3$:

From part (i)(A):

$$x^3-y^3=(x-y)(x^2+xy+y^2)$$

From part (i)(B):

$$x^2+xy+y^2=\left(x+\tfrac{1}{2}y\right)^2+\tfrac{3}{4}y^2$$

Since $\left(x+\frac{1}{2}y\right)^2\geq 0$ and $\frac{3}{4}y^2\geq 0$ for all real x, y, and they are not both zero simultaneously unless $x=y=0$:

In general, $x^2+xy+y^2>0$ for all real x, y (not both zero), and $x^2+xy+y^2=0$ only when $x=y=0$.

If $x>y$, then $x-y>0$, and $x^2+xy+y^2\geq 0$.

More precisely, if $x>y$ then $x\neq y$, so x and y are not both zero simultaneously in a way that makes $x^2+xy+y^2=0$, hence $x^2+xy+y^2>0$.

Therefore:

$$x^3-y^3=\underbrace{(x-y)}_{>0}\cdot\underbrace{(x^2+xy+y^2)}_{>0}>0$$

$$\implies x^3>y^3\square$$

Question 10

Worked Solution

Part (i) — Multiply out $(3^n + 1)(3^n - 1)$:

$$(3^n + 1)(3^n - 1) = (3^n)^2 - 1^2 = 3^{2n} - 1$$

$$(3^n + 1)(3^n - 1) = 3^{2n} - 1$$

Part (ii) — Prove $3^{2n} - 1$ is divisible by 8 for all positive integers n:

From part (i):

$$3^{2n} - 1 = (3^n + 1)(3^n - 1)$$

Since 3 is odd, 3^n is odd for all positive integers n.

Therefore both $3^n + 1$ and $3^n - 1$ are **even**.

Moreover, $3^n - 1$ and $3^n + 1$ are **consecutive even numbers** (they differ by 2).

Among any two consecutive even numbers, one must be divisible by 4 (since every other even number is divisible by 4).

Therefore $(3^n + 1)(3^n - 1)$ is divisible by $2 \times 4 = 8$.

Hence $3^{2n} - 1$ is divisible by 8 for all positive integers n. □

Question 11

Worked Solution

Part (i) — Disprove: "$3^n + 2$ is prime for all integers $n \geq 0$":

We look for a counterexample.

Test $n = 5$:

$$3^5 + 2 = 243 + 2 = 245 = 5 \times 49$$

245 is **not prime**, so this is a counterexample. The statement is false. □

Part (ii) — Prove no power of 3 has final digit 5:

Consider the units digits of successive powers of 3:

n	3^n
1	3
2	9
3	27
4	81
5	243
6	729

The units digits cycle as $3, 9, 7, 1, 3, 9, 7, 1, \ldots$ with period 4.

The digit 5 never appears in this cycle.

Alternatively: any number ending in 5 is divisible by 5. But 3^n is never divisible by 5, since 3 and 5 share no common factors (3 is not divisible by 5, and neither are any of its powers). Therefore 3^n can never end in 5. □

Question 12

Worked Solution

Prove or disprove: “No cube of an integer has 2 as its units digit.”

List the cubes of integers 0–9 (the units digit of n^3 depends only on the units digit of n):

n	n^3	Units digit
0	0	0
1	1	1
2	8	8
3	27	7
4	64	4
5	125	5
6	216	6
7	343	3
8	512	2
9	729	9

When $n = 8$: $8^3 = 512$, which has units digit **2**.

This is a counterexample, so the statement is **false**. □

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