Question 1

Worked Solution

N is an integer not divisible by 3, so N is of the form $N = 3k + 1$ or $N = 3k + 2$ for some integer k.

Case 1: $N = 3k + 1$

$$N^2 = (3k + 1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$$

Setting $p = 3k^2 + 2k$ (which is an integer), we get $N^2 = 3p + 1$. ✓

Case 2: $N = 3k + 2$

$$N^2 = (3k + 2)^2 = 9k^2 + 12k + 4 = 9k^2 + 12k + 3 + 1 = 3(3k^2 + 4k + 1) + 1$$

Setting $p = 3k^2 + 4k + 1$ (which is an integer), we get $N^2 = 3p + 1$. ✓

In both cases N^2 is of the form $3p + 1$ where p is an integer. □

Question 2

Worked Solution

Part (i) — Show $x^2 - 8x + 17 > 0$ for all real x:

Complete the square:

$$x^2 - 8x + 17 = (x-4)^2 - 16 + 17 = (x-4)^2 + 1$$

Since $(x-4)^2 \geq 0$ for all real x, we have:

$$(x-4)^2 + 1 \geq 1 > 0$$

Therefore $x^2 - 8x + 17 > 0$ for all real values of x. $\square$

Part (ii) — "Adding 3 to a number and squaring gives a result greater than the square of the original number":

This statement is **sometimes true**.

Example where it is true: Let $x = 5$.

$$(5+3)^2 = 64 \quad \text{and} \quad 5^2 = 25 \quad \Rightarrow \quad 64 > 25 \checkmark$$

Example where it is false: Let $x = -5$.

$$(-5+3)^2 = (-2)^2 = 4 \quad \text{and} \quad (-5)^2 = 25 \quad \Rightarrow \quad 4 \not> 25$$

Since the statement holds for some values of x but not others, it is **sometimes true**.

The statement is **sometimes true**.

Question 3

Worked Solution

Part (i) P: $y = 3x^5 - 4x^2 + 12x$, Q: $\frac{\mathrm{d}y}{\mathrm{d}x} = 15x^4 - 8x + 12$

Differentiating P gives exactly Q. However, Q being the derivative of P does not imply that P must be $3x^5 - 4x^2 + 12x$ — there could be an added constant of integration (the $+c$). So $P \Rightarrow Q$ but $Q \not\Rightarrow P$.

$P \Rightarrow Q$

Part (ii) P: $x^5 - 32 = 0$ where x is real, Q: $x = 2$

$x^5 = 32$ has exactly one real solution $x = 2$. So $P \Leftrightarrow Q$.

$P \Leftrightarrow Q$

Part (iii) P: $\ln y < 0$, Q: $y < 1$

$\ln y < 0 \Leftrightarrow y < 1$ (for $y > 0$, which is required for $\ln y$ to be defined). So P and Q are equivalent for valid inputs.

$P \Rightarrow Q$

Note: The mark scheme gives $P \Rightarrow Q$ here since $\ln y$ is undefined when $y \leq 0$, so the "$\Leftarrow$" does not hold.

Question 4

Worked Solution

We need to prove that $n^3 + 2$ is not divisible by 8 for any $n \in \mathbb{N}$.

Consider two cases: n odd and n even.

Case 1: n is odd. Write $n = 2k + 1$ for some integer $k \geq 0$.

$$n^3 + 2 = (2k + 1)^3 + 2 = 8k^3 + 12k^2 + 6k + 1 + 2 = 8k^3 + 12k^2 + 6k + 3$$

$$= 8(k^3 + k^2) + 4k(1 + 2k^0) \ldots$$

More directly: $8k^3 + 12k^2 + 6k + 3$. The terms $8k^3$, $12k^2$ and $6k$ contribute an even number, and adding 3 gives an **odd** number. An odd number is never divisible by 8. ✓

Case 2: n is even. Write $n = 2k$ for some integer $k \geq 1$.

$$n^3 + 2 = (2k)^3 + 2 = 8k^3 + 2$$

$8k^3$ is divisible by 8, but $8k^3 + 2$ leaves a remainder of 2 when divided by 8. So $n^3 + 2$ is **not** divisible by 8. ✓

In both cases $n^3 + 2$ is not divisible by 8, so for all $n \in \mathbb{N}$, $n^3 + 2$ is not divisible by 8. □

Question 5

Worked Solution

P and Q are consecutive odd positive integers with $P > Q$. Since they are consecutive odd integers, $P = Q + 2$, so we can write $Q = 2n - 1$ and $P = 2n + 1$ for some positive integer n.

$$P^2 - Q^2 = (2n+1)^2 - (2n-1)^2$$
$$= (4n^2 + 4n + 1) - (4n^2 - 4n + 1)$$
$$= 8n$$

Since $8n$ is a multiple of 8 for any positive integer n, $P^2 - Q^2$ is always a multiple of 8. □

Question 6

Worked Solution

We want to prove that $3\sqrt{2} > 2\sqrt{3}$ without using a calculator.

Method — compare squares:

Since both sides are positive, the inequality $3\sqrt{2} > 2\sqrt{3}$ holds if and only if $(3\sqrt{2})^2 > (2\sqrt{3})^2$.

LHS:

$$(3\sqrt{2})^2 = 9 \times 2 = 18$$

RHS:

$$(2\sqrt{3})^2 = 4 \times 3 = 12$$

Since $18 > 12$, and both original expressions are positive, we conclude:

$$3\sqrt{2} > 2\sqrt{3} \square$$

Question 7

Worked Solution

Arthur uses $p(n) = 2n^2 + 29$ and checks $n = 0$ to $n = 10$, finding all results prime.

Part (a) — Why has Arthur not completed a proof by exhaustion?

Proof by exhaustion requires **every possible case** to be checked. There are infinitely many non-negative integers, so checking only 11 values ($n = 0$ to $n = 10$) does not cover all cases. Arthur has not proved the statement for all non-negative integers.

Part (b) — Use $n = 29$ as a counterexample:

$$p(29) = 2 \times 29^2 + 29 = 2 \times 841 + 29 = 1682 + 29 = 1711$$

Factorise: $1711 = 29 \times 59$.

Since 1711 has factors other than 1 and itself, it is **not prime**. This is a counterexample, so the expression does not always generate prime numbers. □

Part (c) — Prove $p(n) = an^2 + bn + c$ is never prime for all n and all positive integers a, b, c:

Substitute $n = c$:

$$p(c) = ac^2 + bc + c = c(ac + b + 1)$$

<u>Case 1: $c \geq 2$.</u> Then $p(c) = c(ac+b+1)$ is a product of two integers both greater than 1 (since $ac + b + 1 \geq a + b + 1 \geq 3 > 1$). So $p(c)$ is composite — not prime.

<u>Case 2: $c = 1$.</u> Then $p(0) = a(0)^2 + b(0) + 1 = 1$, which is **not prime**.

In both cases, $p(n)$ is not prime for this value of n, so the statement "$p(n)$ is prime for every non-negative integer n" is false for all possible values of a, b and c. □

Question 8

Worked Solution

Part (a) — Prove the statement "m is an odd number greater than $1 \Rightarrow m^2 + 4$ is prime" is not true:

We need a counterexample — an odd $m > 1$ for which $m^2 + 4$ is not prime.

Try $m = 9$:

$$m^2 + 4 = 81 + 4 = 85 = 5 \times 17$$

85 is not prime. This is a counterexample, so the statement is false. □

Part (b) — Prove: n is a positive integer $\Rightarrow n^2 + 1$ is not a multiple of 4:

Consider two cases based on the parity of n.

Case 1: n is odd. Write $n = 2k + 1$ for some integer $k \geq 0$.

$$n^2 + 1 = (2k + 1)^2 + 1 = 4k^2 + 4k + 1 + 1 = 4k^2 + 4k + 2 = 4(k^2 + k) + 2$$

This is of the form $4 \times \text{integer} + 2$, so it leaves remainder 2 when divided by 4. Therefore it is **not a multiple of 4**. ✓

Case 2: n is even. Write $n = 2k$ for some integer $k \geq 1$.

$$n^2 + 1 = (2k)^2 + 1 = 4k^2 + 1$$

This is of the form $4 \times \text{integer} + 1$, so it leaves remainder 1 when divided by 4. Therefore it is **not a multiple of 4**. ✓

In both cases $n^2 + 1$ is not a multiple of 4, so the statement is proved for all positive integers n. □

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