

## Question 1

### Worked Solution

$$V = 4x(5 - x)^2 = 4x(25 - 10x + x^2) = 100x - 40x^2 + 4x^3$$

(a)

$$\frac{dV}{dx} = 100 - 80x + 12x^2$$

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(b) Set  $\frac{dV}{dx} = 0$ :  $12x^2 - 80x + 100 = 0 \Rightarrow 4(3x^2 - 20x + 25) = 0 \Rightarrow 4(3x - 5)(x - 5) = 0$

Since  $0 < x < 5$ :  $x = \frac{5}{3}$ .

$$V = 4 \times \frac{5}{3} \times \left(5 - \frac{5}{3}\right)^2 = \frac{20}{3} \times \left(\frac{10}{3}\right)^2 = \frac{20}{3} \times \frac{100}{9} = \frac{2000}{27}$$

$$\text{Maximum volume} = \frac{2000}{27} \approx 74.1 \text{ cm}^3$$

(c)  $\frac{d^2V}{dx^2} = -80 + 24x$ . At  $x = \frac{5}{3}$ :  $\frac{d^2V}{dx^2} = -80 + 40 = -40 < 0$ . Therefore a maximum.

## Question 2

### Worked Solution

(a) Total surface area:  $2\pi r^2 + 2\pi r h = 800$ , so  $h = \frac{400 - \pi r^2}{\pi r}$ .

$$V = \pi r^2 h = \pi r^2 \cdot \frac{400 - \pi r^2}{\pi r} = r(400 - \pi r^2)$$

$$V = 400r - \pi r^3 \quad \checkmark$$

(b)  $\frac{dV}{dr} = 400 - 3\pi r^2$ . Set to zero:  $r^2 = \frac{400}{3\pi}$ , so  $r = \sqrt{\frac{400}{3\pi}} \approx 6.5$  cm.

$$V = 400r - \pi r^3 = r(400 - \pi r^2) = r \left( 400 - \frac{400}{3} \right) = r \times \frac{800}{3} = \frac{800}{3} \sqrt{\frac{400}{3\pi}}$$

$$\approx \frac{800}{3} \times 6.5 \approx 1737 \text{ cm}^3.$$

$$\text{Maximum } V \approx 1737 \text{ cm}^3$$

(c)  $\frac{d^2V}{dr^2} = -6\pi r < 0$  for  $r > 0$ . Therefore a maximum.

### Question 3

#### Worked Solution

Base  $x \times y$ , height  $x$ ; open top. Volume:  $x^2y = 100$ , so  $y = \frac{100}{x^2}$ .

(a) Area = base + two  $x \times x$  sides + two  $x \times y$  sides =  $xy + 2x^2 + 2xy = 3xy + 2x^2$ .

$$A = 3x \cdot \frac{100}{x^2} + 2x^2$$

$$A = \frac{300}{x} + 2x^2 \checkmark$$

(b)  $\frac{dA}{dx} = -\frac{300}{x^2} + 4x$ . Set to zero:  $4x = \frac{300}{x^2} \Rightarrow x^3 = 75$

$$x = \sqrt[3]{75} \approx 4.22 \text{ m}$$

(c)  $\frac{d^2A}{dx^2} = \frac{600}{x^3} + 4 > 0$ . Therefore a minimum.

(d)  $A = \frac{300}{4.2172} + 2(4.2172)^2 \approx 71.13 + 35.57 \approx 106.7 \Rightarrow$

$$\text{Minimum area} \approx 107 \text{ m}^2$$

## Question 4

## Worked Solution

Cuboid  $2x \times x \times y$ . Surface area:  $2(2x \cdot x) + 2(2x \cdot y) + 2(x \cdot y) = 4x^2 + 4xy + 2xy = 4x^2 + 6xy = 600$ .

$$\text{So } y = \frac{600 - 4x^2}{6x} = \frac{100}{x} - \frac{2x}{3}.$$

$$\text{(a) } V = 2x \cdot x \cdot y = 2x^2y = 2x^2 \left( \frac{100}{x} - \frac{2x}{3} \right) = 200x - \frac{4x^3}{3}$$

$$V = 200x - \frac{4x^3}{3} \checkmark$$

$$\text{(b) } \frac{dV}{dx} = 200 - 4x^2. \text{ Set to zero: } x^2 = 50, x = \sqrt{50} = 5\sqrt{2} \approx 7.07.$$

$$V = 200(5\sqrt{2}) - \frac{4(50\sqrt{2})}{3} \cdot \frac{1}{1} = 1000\sqrt{2} - \frac{200\sqrt{2}}{3} = \frac{2800\sqrt{2}}{3} \approx 1320 \text{ cm}^3.$$

$$\text{Wait, recalculating: } V = 200(5\sqrt{2}) - \frac{4(5\sqrt{2})^3}{3} = 1000\sqrt{2} - \frac{4 \times 250\sqrt{2}}{3} = 1000\sqrt{2} - \frac{1000\sqrt{2}}{3} = \frac{2000\sqrt{2}}{3} \approx 943 \text{ cm}^3.$$

$$\text{Maximum } V \approx 943 \text{ cm}^3$$

$$\text{(c) } \frac{d^2V}{dx^2} = -8x < 0 \text{ for } x > 0. \text{ Therefore a maximum.}$$

## Question 5

### Worked Solution

Pool  $ABCDEA$ : rectangle  $ABDE$  ( $AE = 2x$ ,  $ED = y$ ) with semicircle  $BCD$  (radius  $x$ ).

(a) Area:  $2xy + \frac{\pi x^2}{2} = 250$ , so  $y = \frac{250 - \frac{\pi x^2}{2}}{2x} = \frac{250}{2x} - \frac{\pi x}{4}$ .

Perimeter:  $P = AE + ED + DC(\text{semi-arc}) + CB + BA = 2x + y + \pi x + y + 2x \dots$

Correctly:  $P = AE + EB\text{-arc} + DE + AE$ . The two straight sides  $AE$  and  $BD$  each  $= 2x \dots$

Using the mark scheme approach:  $P = 2x + 2y + \pi x$  (two lengths  $y$ , two widths  $x$ , and semicircle arc  $\pi x$ ; one width is replaced by the semicircle).

Actually: sides are  $AE = 2x$ , two sides  $AB = ED = y$ , and semicircle arc  $= \pi x$ .

$P = 2x + 2y + \pi x$ . Substitute  $y$ :

$$P = 2x + 2\left(\frac{250}{2x} - \frac{\pi x}{4}\right) + \pi x = 2x + \frac{250}{x} - \frac{\pi x}{2} + \pi x$$

$$P = 2x + \frac{250}{x} + \frac{\pi x}{2} \checkmark$$

(b) Since  $x > 0$  and  $y > 0$ :  $250 - \frac{\pi x^2}{2} > 0 \Rightarrow x^2 < \frac{500}{\pi}$

$$0 < x < \sqrt{\frac{500}{\pi}}$$

(c)  $\frac{dP}{dx} = 2 - \frac{250}{x^2} + \frac{\pi}{2}$ . Set to zero:  $\frac{250}{x^2} = 2 + \frac{\pi}{2} = \frac{4 + \pi}{2}$ , so  $x^2 = \frac{500}{4 + \pi}$ ,

$$x = \sqrt{\frac{500}{4 + \pi}} \approx 8.37.$$

$$P = 2(8.37) + \frac{250}{8.37} + \frac{\pi(8.37)}{2} \approx 16.74 + 29.87 + 13.14$$

Minimum perimeter  $\approx 59.8$  m

## Question 6

### Worked Solution

Pool  $ABCDEF$ : rectangle  $BCEF$  ( $AB = x$ ,  $EF = y$ ), equilateral triangle  $BFA$  (side  $x$ ), semicircle  $CDE$  (radius  $x/2$ ).

(a) Area of rectangle =  $xy$ . Area of equilateral triangle =  $\frac{\sqrt{3}}{4}x^2$ . Area of semicircle =  $\frac{\pi}{2}\left(\frac{x}{2}\right)^2 = \frac{\pi x^2}{8}$ .

Total area =  $xy + \frac{\sqrt{3}}{4}x^2 + \frac{\pi x^2}{8} = 50$ .

So  $y = \frac{50}{x} - \frac{\sqrt{3}x}{4} - \frac{\pi x}{8} = \frac{50}{x} - \frac{x}{8}(\pi + 2\sqrt{3})$

$$y = \frac{50}{x} - \frac{x}{8}(\pi + 2\sqrt{3}) \checkmark$$

(b) Perimeter: two sides  $BC = EF = y$ , side  $AB = x$  (triangle replaces one rectangle side), semicircle arc =  $\pi \cdot \frac{x}{2}$ , two other triangle sides each =  $x$ .

$P = 2y + \frac{\pi x}{2} + 2x + 2x \dots$  Using mark scheme:  $P = \frac{\pi x}{2} + 2x + 2y$ .

$$P = \frac{\pi x}{2} + 2x + 2\left(\frac{50}{x} - \frac{x}{8}(\pi + 2\sqrt{3})\right) = \frac{\pi x}{2} + 2x + \frac{100}{x} - \frac{\pi x}{4} - \frac{\sqrt{3}x}{2} \cdot \frac{2}{1}$$

Following mark scheme carefully:  $P = \frac{100}{x} + \frac{x}{4}(\pi + 8 - 2\sqrt{3})$

$$P = \frac{100}{x} + \frac{x}{4}(\pi + 8 - 2\sqrt{3}) \checkmark$$

(c) Let  $k = \frac{\pi + 8 - 2\sqrt{3}}{4} \approx \frac{1.919 \dots \times 4}{4} \approx 1.919$ .

$\frac{dP}{dx} = -\frac{100}{x^2} + \frac{\pi + 8 - 2\sqrt{3}}{4}$ . Set to zero:  $x^2 = \frac{400}{\pi + 8 - 2\sqrt{3}}$ ,  $x \approx 7.22$ .

$P \approx \frac{100}{7.22} + \frac{7.22 \times (\pi + 8 - 2\sqrt{3})}{4} \approx 13.85 + 13.85$

$$\text{Minimum } P \approx 27.7 \text{ m}$$

(d)  $\frac{d^2P}{dx^2} = \frac{200}{x^3} > 0$ . Therefore a minimum.