Practical Applications of Differentiation Exam Questions Sheet 2

Q1.

Question Number	Scheme		Marks
(a)	$V = 4x(5-x)^2 = 4x(25-10x+x^2)$		
	So, $V = 100x - 40x^2 + 4x^3$	$\pm\alpha x \pm \beta x^2 \pm \gamma x^3$, where $\alpha, \beta, \gamma \neq 0$	M1
		$V = 100x - 40x^2 + 4x^3$	A1
	$\frac{dV}{dx} = 100 - 80x + 12x^2$	At least two of their expanded terms differentiated correctly.	M1
		$100 - 80x + 12x^2$	A1 cao
			(4)
(b)	$100 - 80x + 12x^2 = 0$	Sets their $\frac{dV}{dx}$ from part (a) $= 0$	M1
	$\{\Rightarrow 4(3x^2 - 20x + 25) = 0 \Rightarrow 4(3x-5)(x-5) = 0\}$		
	$\{\text{As } 0 < x < 5\}$ $x = \frac{5}{3}$	$x = \frac{5}{3}$ or $x =$ awrt 1.67	A1
	$x = \frac{5}{3},\ V = 4(\frac{5}{3})(5 - \frac{5}{3})^2$	Substitute candidate's value of x **where** $0 < x < 5$ into a formula for V.	dM1
	So, $V = \frac{2000}{27} = 74\frac{2}{27} = 74.074...$	Either $\frac{2000}{27}$ or $74\frac{2}{27}$ or awrt 74.1	A1
			(4)
(c)	$\frac{d^2V}{dx^2} = -80 + 24x$	Differentiates their $\frac{dV}{dx}$ **correctly** to give $\frac{d^2V}{dx^2}$.	M1
	When $x = \frac{5}{3}$, $\frac{d^2V}{dx^2} = -80 + 24\left(\frac{5}{3}\right)$		
	$\frac{d^2V}{dx^2} = -40 < 0 \Rightarrow V$ is a maximum	$\frac{d^2V}{dx^2} = -40$ and < 0 or negative and maximum.	A1 cso
			(2)
			[10]

	Notes
(a)	1st M1 for a three term cubic in the form $\pm\alpha x \pm \beta x^2 \pm \gamma x^3$. Note that an un-combined $\pm\alpha x \pm \lambda x^2 \pm \mu x^2 \pm \gamma x^3$, $\alpha, \lambda, \mu, \gamma \neq 0$ is fine for the 1st M1. 1st A1 for either $100x - 40x^2 + 4x^3$ or $100x - 20x^2 - 20x^2 + 4x^3$. 2nd M1 for any two of their expanded terms differentiated correctly. NB: If expanded expression is divided by a constant, then the 2nd M1 can be awarded for at least two terms are correct. Note for un-combined $\pm\lambda x^2 \pm \mu x^2$, $\pm 2\lambda x \pm 2\mu x$ counts as one term differentiated correctly. 2nd A1 for $100 - 80x + 12x^2$, **cao**. **Note:** See appendix for those candidates who apply the product rule of differentiation.

Q2.

Question Number	Scheme	Marks
(a)	$2\pi rh + 2\pi r^2 = 800$	B1
	$h = \frac{400-\pi r^2}{\pi r}$, $V = \pi r^2\left(\frac{400-\pi r^2}{\pi r}\right) = 400r - \pi r^3$ (*)	M1, M1 A1 (4)
(b)	$\frac{dV}{dr} = 400 - 3\pi r^2$	M1 A1
	$400 - 3\pi\ r^2 = 0$ $r^2 = ...,$ $r = \sqrt{\frac{400}{3\pi}}$ $(= 6.5\ (2\text{ s.f.}))$	M1 A1
	$V = 400r - \pi r^3 = 1737 = \frac{800}{3}\sqrt{\frac{400}{3\pi}}\ (\text{cm}^3)$ (accept awrt 1737 or exact answer)	M1 A1 (6)
(c)	$\frac{d^2V}{dr^2} = -6\pi r$, Negative, $\therefore$ maximum (Parts (b) and (c) should be considered together when marking)	M1 A1 (2) [12]
Other methods for part (c):	Either: M: Find value of $\frac{dV}{dr}$ on each side of "$r = \sqrt{\frac{400}{3\pi}}$" and consider sign. A: Indicate sign change of positive to negative for $\frac{dV}{dr}$, and conclude max. Or: M: Find value of V on each side of "$r = \sqrt{\frac{400}{3\pi}}$" and compare with "1737". A: Indicate that both values are less than 1737 or 1737.25, and conclude max.	
Notes (a)	B1: For any correct form of this equation (may be unsimplified, may be implied by 1st M1) M1 : Making h the subject of their three or four term formula M1: Substituting expression for h into $\pi r^2 h$ (independent mark) Must now be expression in r only. A1: cso	
(b)	M1: At least one power of r decreased by 1 A1: cao M1: Setting $\frac{dV}{dr}$ =0 and finding a value for correct power of r for candidate A1 : This mark may be credited if the value of V is correct. Otherwise answers should round to 6.5 (allow ± 6.5)or be exact answer M1: Substitute a positive value of r to give V A1: 1737 or 1737.25….. or exact answer	

(c)	M1: needs complete method e.g.attempts differentiation (power reduced) of their first derivative and considers its sign A1(first method) should be $-6\pi r$ (do not need to substitute r and can condone wrong r if found in (b)) Need to conclude maximum or indicate by a tick that it is maximum. Throughout allow confused notation such as dy/dx for dV/dr	
Alternative for (a)	$A = 2\pi r^2 + 2\pi rh$, $\frac{4}{2} \times r = \pi r^3 + \pi r^2 h$ is M1 Equate to $400r$ B1 Then $V = 400r - \pi r^3$ is M1 A1	

Q3.

Question Number	Scheme	Marks
(a)	(Total area) = $3xy + 2x^2$	B1
	(Vol:) $x^2y = 100$ $\left(y = \frac{100}{x^2}, xy = \frac{100}{x}\right)$	B1
(b)	Deriving expression for area in terms of x only	M1
	(Substitution, or clear use of, y or xy into expression for area)	
	(Area =) $\frac{300}{x} + 2x^2$ **AG**	A1 cso (4)
(c)	$\frac{dA}{dx} = -\frac{300}{x^2} + 4x$	M1A1
	Setting $\frac{dA}{dx} = 0$ and finding a value for correct power of x, for cand. M1	
	[$x^3 = 75$]	
	$x = 4.2172$ awrt 4.22 (allow exact $\sqrt[3]{75}$)	A1 (4)
	$\frac{d^2A}{dx^2} = \frac{600}{x^3} + 4$ = positive , > 0; therefore minimum	M1;A1 (2)
(d)	Substituting found value of x into (a)	M1
	(Or finding y for found x and substituting both in $3xy + 2x^2$)	
	$\left[y = \frac{100}{4.2172^2} = 5.6228\right]$	
	Area = 106.707 awrt 107	A1 (2)
		[12]

Notes		
	(a) First B1: Earned for correct unsimplified expression, isw. (b) First M1: At least one power of x decreased by 1, and no "c" term. (c) For M1: Find $\frac{d^2A}{dx^2}$ and explicitly consider its sign, state > 0 or "positive" A1: Candidate's $\frac{d^2A}{dx^2}$ must be correct for their $\frac{dA}{dx}$, sign must be + ve and conclusion "so minimum", (allow QED, √). (may be wrong x, or even no value of x found) <u>Alternative:</u> M1: Find value of $\frac{dA}{dx}$ on either side of "$x = \sqrt[3]{75}$" and consider sign A1: Indicate sign change of negative to positive for $\frac{dA}{dx}$, and conclude minimum. OR M1: Consider values of A on either side of "$x = \sqrt[3]{75}$" and compare with"107" A1: Both values greater than "$x = 107$" and conclude minimum. Allow marks for (c) and (d) where seen; even if part labelling confused. Throughout , allow confused notation, such as dy/dx for dA/dx.	

Q4.

Question number	Scheme	Marks
	(a) $4x^2 + 6xy = 600$	M1 A1
	$V = 2x^2y = 2x^2\left(\frac{600-4x^2}{6x}\right)$ $\quad V = 200x - \frac{4x^3}{3}$ $\quad$ (*)	M1 A1cso (4)
	(b) $\frac{dV}{dx} = 200 - 4x^2$	B1
	Equate their $\frac{dV}{dx}$ to 0 and solve for x^2 or x : $x^2 = 50$ or $x = \sqrt{50}$ (7.07…)	M1 A1
	Evaluate V: $\quad V = 200(\sqrt{50}) - \frac{4}{3}(50\sqrt{50}) = 943 \text{ cm}^3$ $\quad$ Allow awrt	M1 A1 (5)
	(c) $\frac{d^2V}{dx^2} = -8x$ Negative, $\quad \therefore$ Maximum	M1, A1ft (2)
		11

(a) 1st M: Attempting an expression in terms of x and y for the total surface area (the expression should be dimensionally correct).

1st A: Correct expression (not necessarily simplified), equated to 600.

2nd M: Substituting their y into $2x^2y$ to form an expression in terms of x only. (Or substituting y from $2x^2y$ into their area equation).

(b) 1st A: Ignore $x = -\sqrt{50}$, if seen.

The 2nd M mark (for substituting their x value into the given expression for V) is dependent on the 1st M.

Final A: Allow also exact value $\frac{400\sqrt{50}}{3}$ or $\frac{2000\sqrt{2}}{3}$ or equiv. single term.

(c) Allow marks if the work for (c) is seen in (b) (or vice-versa).

M: Find second derivative and consider its sign.

A: Second derivative following through correctly from their $\frac{dV}{dx}$, and correct reason/conclusion (it must be a maximum, not a minimum).
An actual value of x does not have to be used… this mark can still be awarded if no x value has been found or if a wrong x value is used.

Alternative:

M: Find value of $\frac{dV}{dx}$ on each side of "$x = \sqrt{50}$" and consider sign.

A: Indicate sign change of positive to negative for $\frac{dV}{dx}$, and conclude max.

Alternative:

M: Find value of V on each side of "$x = \sqrt{50}$" and compare with "943".
A: Indicate that both values are less than 943, and conclude max.

Q5.

Question	Scheme	Marks	AOs
(a)	Sets $2xy+\frac{\pi x^2}{2}=250$	B1	2.1
	Obtain $y=\frac{250-\frac{\pi x^2}{2}}{2x}$ and substitute into P	M1	1.1b
	Use $P=2x+2y+\pi x$ with their y substituted	M1	2.1
	$P=2x+\frac{250}{x}-\frac{\pi x^2}{2x}+\pi x=2x+\frac{250}{x}+\frac{\pi x}{2}$ *	A1*	1.1b
		(4)	
(b)	$x>0$ and $y>0$ (distance) $\Rightarrow \frac{250-\frac{\pi x^2}{2}}{2x}>0$ or $250-\frac{\pi x^2}{2}>0$ o.e.	M1	2.4
	As x and y are distances they are positive so $0<x<\sqrt{\frac{500}{\pi}}$ *	A1*	3.2a
		(2)	
(c)	Differentiates P with negative index correct in $\frac{dP}{dx}$; $x^{-1}\to x^{-2}$	M1	3.4
	$\frac{dP}{dx}=2-\frac{250}{x^2}+\frac{\pi}{2}$	A1	1.1b
	Sets $\frac{dP}{dx}=0$ and proceeds to $x=$	M1	1.1b
	Substitutes their x into $P=2x+\frac{250}{x}+\frac{\pi x}{2}$ to give perimeter = 59.8m.	A1	1.1b
		(4)	
			(10 marks)

Notes

(a) B1 : Correct area equation

M1 : Rearranges **their** area equation to make y the subject of the formula and attempt to use with an expression for P

M1 : Use correct equation for perimeter with their y substituted

A1*: Completely correct solution to obtain and state printed answer

(b) M1 : States $x>0$ and $y>0$ and uses their expression from (a) to form inequality

A1*: Explains that x and y are positive because they are distances, and uses correct expression for y to give the printed answer correctly.

(c) M1: Attempt to differentiate P (deals with negative power of x correctly)

A1 : Correct differentiation

M1 : Sets derived function equal to zero and obtains $x=$

A1: The value of x may not be seen (it is 8.37 to 3sf or $\sqrt{\left(\frac{500}{4+\pi}\right)}$).

Need to see awrt 59.8m with units included for the perimeter.

Q6.

Question Number	Scheme		Marks
(a)	$\{A=\}\ xy+\frac{\pi}{2}\left(\frac{x}{2}\right)^2+\frac{1}{2}x^2\sin 60^{\circ}$	M1: An attempt to find 3 areas of the form: $xy,\ \ p\pi x^2$ and qx^2 A1: Correct expression for A (terms must be added)	M1A1
	$50=xy+\frac{\pi x^2}{8}+\frac{\sqrt{3}x^2}{4}\Rightarrow y=\frac{50}{x}-\frac{\pi x}{8}-\frac{\sqrt{3}x}{4}\Rightarrow \underline{y=\frac{50}{x}-\frac{x}{8}\left(\pi+2\sqrt{3}\right)}$ * Correct proof with no errors seen		A1 *
			[3]
(b)	$\{P=\}\ \frac{\pi x}{2}+2x+2y$	Correct expression for P in terms of x and y	B1
	$P=\frac{\pi x}{2}+2x+2\left(\frac{50}{x}-\frac{x}{8}\left(\pi+2\sqrt{3}\right)\right)$	Substitutes the given expression for y into an expression for P where P is at least of the form $\alpha x+\beta y$	M1
	$P=\frac{\pi x}{2}+2x+\frac{100}{x}-\frac{\pi x}{4}-\frac{\sqrt{3}}{2}x\Rightarrow P=\frac{100}{x}+\frac{\pi x}{4}+2x-\frac{\sqrt{3}}{2}x$		
	$\Rightarrow \underline{P=\frac{100}{x}+\frac{x}{4}\left(\pi+8-2\sqrt{3}\right)}$	Correct proof with no errors seen	A1 *
			[3]
	(Note $\frac{\pi+8-2\sqrt{3}}{4}=1.919......$)		
(c) and (d) can be marked together	$\frac{dP}{dx}=-100x^{-2}+\frac{\pi+8-2\sqrt{3}}{4}$	M1: Either $\mu x\to\mu$ or $\frac{100}{x}\to\frac{\pm\lambda}{x^2}$ A1: Correct differentiation (need not be simplified). Allow $-100x^{-2}+(\text{awrt}1{\cdot}92)$	M1A1
	$-100x^{-2}+\frac{\pi+8-2\sqrt{3}}{4}=0\Rightarrow x=...$	Their $P'=0$ and attempt to solve as far as $x=....$ (ignore poor manipulation)	M1
	$\Rightarrow x=\sqrt{\frac{400}{\pi+8-2\sqrt{3}}}=7.2180574...$	$\sqrt{\frac{400}{\pi+8-2\sqrt{3}}}$ or awrt 7.2 **and no other values**	A1
	$\{x=7.218...\}\ \Rightarrow P=27.708...$ (m)	awrt 27.7	A1
			[5]
	$\frac{d^2P}{dx^2}=\frac{200}{x^3}>0\Rightarrow$ Minimum	M1: Finds P'' ($x^n\to x^{n-1}$ allow for constant $\to 0$) **and considers sign** A1ft: $\frac{200}{x^3}$ (need not be simplified) and >0 and conclusion. Only follow through on a correct P'' and a single positive value of x found earlier.	M1A1ft
			[2]
			Total 13