Question 1 (Jan 2006, Q8i)

Worked Solution

$f(x) = 2x^3 + kx^2 - x + 6$, and $(x+1)$ is a factor.

Show $k = -5$ and factorise $f(x)$ completely.

Since $(x+1)$ is a factor, $f(-1) = 0$:

$$2(-1)^3 + k(-1)^2 - (-1) + 6 = 0$$

$$-2 + k + 1 + 6 = 0 \implies k + 5 = 0 \implies k = -5 \quad \square$$

So $f(x) = 2x^3 - 5x^2 - x + 6$. Divide by $(x+1)$.

Method 1 — Long division:

$$\begin{array}{r|l} & 2x^2 - 7x + 6 \\ \hline x+1 & 2x^3 - 5x^2 - x + 6 \\ & 2x^3 + 2x^2 \\ \hline & -7x^2 - x \\ & -7x^2 - 7x \\ \hline & 6x + 6 \\ & 6x + 6 \\ \hline & 0 \end{array}$$

Method 2 — Grid method:

Seek $(x+1)(2x^2 + bx + c)$:

$\times$	$2x^2$	bx	c
x	$2x^3$	bx^2	cx
$+1$	$2x^2$	bx	c

Match coefficients:

- x^2: $b + 2 = -5 \Rightarrow b = -7$
- x^0: $c = 6$
- Check x: $c + b = 6 - 7 = -1$ ✓

Quotient is $2x^2 - 7x + 6$. Factorise: $(2x-3)(x-2)$.

$f(x) = (x+1)(2x-3)(x-2)$

Question 2 (Jan 2007, Q8)

Worked Solution

$f(x) = x^3 - 9x^2 + 7x + 33$

Part (i): Remainder when divided by $(x + 2)$.

$f(-2) = (-8) - 9(4) + 7(-2) + 33 = -8 - 36 - 14 + 33 = -25$

Remainder $= -25$

Part (ii): Show $(x - 3)$ is a factor.

$f(3) = 27 - 9(9) + 7(3) + 33 = 27 - 81 + 21 + 33 = 0$ □

Part (iii): Solve $f(x) = 0$.

Divide by $(x - 3)$.

Method 1 — Long division:

$$\begin{array}{r} x^2 - 6x - 11 \\ x - 3 \,\big)\, x^3 - 9x^2 + 7x + 33 \\ x^3 - 3x^2 \\ \hline -6x^2 + 7x \\ -6x^2 + 18x \\ \hline -11x + 33 \\ -11x + 33 \\ \hline 0 \end{array}$$

Method 2 — Grid method:

Seek $(x - 3)(x^2 + bx + c)$:

$\times$	x^2	bx	c
x	x^3	bx^2	cx
-3	$-3x^2$	$-3bx$	$-3c$

Match coefficients:

- x^2: $b - 3 = -9 \Rightarrow b = -6$
- x^0: $-3c = 33 \Rightarrow c = -11$
- Check x: $c - 3b = -11 + 18 = 7$ ✓

So $f(x) = (x - 3)(x^2 - 6x - 11)$.

The quadratic $x^2 - 6x - 11$ has no integer roots, so use the quadratic formula:

$$x = \frac{6 \pm \sqrt{36 + 44}}{2} = \frac{6 \pm \sqrt{80}}{2} = 3 \pm 2\sqrt{5}$$

$x = 3, \quad x = 3 + 2\sqrt{5}, \quad x = 3 - 2\sqrt{5}$

Question 3 (Jun 2007, Q9i)

Worked Solution

$f(x) = x^3 + 6x^2 + x - 4$

Part (i)(a): Show $(x + 1)$ is a factor.

$f(-1) = -1 + 6 - 1 - 4 = 0$ □

Part (i)(b): Find the exact roots of $f(x) = 0$.

Divide by $(x + 1)$.

Method 1 — Long division:

$$\begin{array}{r|l} & x^2 + 5x - 4 \\ x+1 & x^3 + 6x^2 + x - 4 \\ & x^3 + x^2 \\ & 5x^2 + x \\ & 5x^2 + 5x \\ & -4x - 4 \\ & -4x - 4 \\ & 0 \end{array}$$

Method 2 — Grid method:

Seek $(x + 1)(x^2 + bx + c)$:

$\times$	x^2	bx	c
x	x^3	bx^2	cx
$+1$	x^2	bx	c

Match coefficients:

- x^2: $b + 1 = 6 \Rightarrow b = 5$
- x^0: $c = -4$
- Check x: $c + b = -4 + 5 = 1$ ✓

$f(x) = (x + 1)(x^2 + 5x - 4)$. Apply quadratic formula to $x^2 + 5x - 4 = 0$:

$$x = \frac{-5 \pm \sqrt{25 + 16}}{2} = \frac{-5 \pm \sqrt{41}}{2}$$

$$x = -1, \quad x = \frac{-5 + \sqrt{41}}{2}, \quad x = \frac{-5 - \sqrt{41}}{2}$$

Question 4 (Jun 2008, Q4)

Worked Solution

$f(x) = ax^3 - 4x^2 - 7ax + 12$

Part (i): Given $(x-3)$ is a factor, find a.

$f(3) = 0$:

$$27a - 36 - 21a + 12 = 0 \implies 6a - 24 = 0 \implies a = 4$$

$a = 4$

Part (ii): Find the remainder when $f(x) = 4x^3 - 4x^2 - 28x + 12$ is divided by $(x+2)$.

$f(-2) = 4(-8) - 4(4) - 28(-2) + 12 = -32 - 16 + 56 + 12 = 20$

Remainder $= 20$

Question 5 (Jun 2009, Q7)

Worked Solution

$f(x) = 2x^3 + 9x^2 + 11x - 8$

Part (i): Remainder when divided by $(x + 2)$.

$f(-2) = 2(-8) + 9(4) + 11(-2) - 8 = -16 + 36 - 22 - 8 = -10$

Remainder $= -10$

Part (ii): Show $(2x - 1)$ is a factor.

Substitute $x = \frac{1}{2}$:

$$f\left(\tfrac{1}{2}\right) = 2\left(\tfrac{1}{8}\right) + 9\left(\tfrac{1}{4}\right) + 11\left(\tfrac{1}{2}\right) - 8 = \tfrac{1}{4} + \tfrac{9}{4} + \tfrac{11}{2} - 8 = \tfrac{1+9}{4} + \tfrac{22}{4} - \tfrac{32}{4} = \tfrac{32-32}{4} = 0 \quad \square$$

Part (iii): Express $f(x)$ as a product of a linear and a quadratic factor.

Divide by $(2x - 1)$.

Method 1 — Long division:

$$\begin{array}{r} x^2 + 5x + 8 \\ 2x - 1 \overline{)\, 2x^3 + 9x^2 + 11x - 8} \\ \underline{2x^3 - x^2} \\ 10x^2 + 11x \\ \underline{10x^2 - 5x} \\ 16x - 8 \\ \underline{16x - 8} \\ 0 \end{array}$$

Method 2 — Grid method:

Seek $(2x - 1)(x^2 + bx + c)$:

$\times$	x^2	bx	c
$2x$	$2x^3$	$2bx^2$	$2cx$
-1	$-x^2$	$-bx$	$-c$

Match coefficients:

- x^2: $2b - 1 = 9 \Rightarrow b = 5$
- x^0: $-c = -8 \Rightarrow c = 8$
- Check x: $2c - b = 16 - 5 = 11$ ✓

$f(x) = (2x - 1)(x^2 + 5x + 8)$

Part (iv): Number of real roots.

The discriminant of $x^2 + 5x + 8$ is $25 - 32 = -7 < 0$, so the quadratic has no real roots. Therefore $f(x) = 0$ has exactly one real root, $x = \frac{1}{2}$.

One real root (the quadratic factor has negative discriminant)

Question 6 (Jan 2012, Q5)

Worked Solution

$f(x) = 2x^3 + 3x^2 - 17x + 6$

Part (i): Remainder when divided by $(x - 3)$.

$f(3) = 2(27) + 3(9) - 17(3) + 6 = 54 + 27 - 51 + 6 = 36$

Remainder $= 36$

Part (ii): Given $f(2) = 0$, express $f(x)$ as a product of a linear and quadratic factor.

Divide by $(x - 2)$.

Method 1 — Long division:

$$\begin{array}{r} 2x^2 + 7x - 3 \\ x - 2 \,\big)\, 2x^3 + 3x^2 - 17x + 6 \\ 2x^3 - 4x^2 \\ \hline 7x^2 - 17x \\ 7x^2 - 14x \\ \hline -3x + 6 \\ -3x + 6 \\ \hline 0 \end{array}$$

Method 2 — Grid method:

Seek $(x - 2)(2x^2 + bx + c)$:

$\times$	$2x^2$	bx	c
x	$2x^3$	bx^2	cx
-2	$-4x^2$	$-2bx$	$-2c$

Match coefficients:

- x^2: $b - 4 = 3 \Rightarrow b = 7$
- x^0: $-2c = 6 \Rightarrow c = -3$
- Check x: $c - 2b = -3 - 14 = -17$ ✓

$f(x) = (x - 2)(2x^2 + 7x - 3)$

Part (iii): Number of real roots.

Discriminant of $2x^2 + 7x - 3$: $\Delta = 49 + 24 = 73 > 0$, so two distinct real roots from the quadratic. Together with $x = 2$, the equation $f(x) = 0$ has 3 real roots.

3 real roots (discriminant of quadratic $= 73 > 0$)

Question 7 (Jun 2013, Q9)

Worked Solution

$f(x) = 4x^3 - 7x - 3$

Part (i): Remainder when divided by $(x - 2)$.

$f(2) = 4(8) - 7(2) - 3 = 32 - 14 - 3 = 15$

Remainder $= 15$

Part (ii): Show $(2x + 1)$ is a factor and factorise completely.

Substitute $x = -\frac{1}{2}$:

$$f\left(-\tfrac{1}{2}\right) = 4\left(-\tfrac{1}{8}\right) - 7\left(-\tfrac{1}{2}\right) - 3 = -\tfrac{1}{2} + \tfrac{7}{2} - 3 = 3 - 3 = 0 \quad \square$$

Divide $4x^3 - 7x - 3$ by $(2x + 1)$.

Method 1 — Long division:

$$\begin{array}{r} 2x^2 - x - 3 \\ 2x+1 \;\overline{\big)\; 4x^3 + 0x^2 - 7x - 3} \\ \underline{4x^3 + 2x^2} \\ -2x^2 - 7x \\ \underline{-2x^2 - x} \\ -6x - 3 \\ \underline{-6x - 3} \\ 0 \end{array}$$

Method 2 — Grid method:

Seek $(2x + 1)(2x^2 + bx + c)$:

$\times$	$2x^2$	bx	c
$2x$	$4x^3$	$2bx^2$	$2cx$
$+1$	$2x^2$	bx	c

Match coefficients:

- x^2: $2b + 2 = 0 \Rightarrow b = -1$
- x^0: $c = -3$
- Check x: $2c + b = -6 - 1 = -7$ ✓

$f(x) = (2x + 1)(2x^2 - x - 3) = (2x + 1)(2x - 3)(x + 1)$.

$f(x) = (2x + 1)(2x - 3)(x + 1)$

Part (iii): Solve $4\cos^3\theta - 7\cos\theta - 3 = 0$ for $0° \le \theta \le 360°$.

This is $f(\cos\theta) = 0$, so $\cos\theta = -\frac{1}{2}$, $\cos\theta = \frac{3}{2}$ (impossible), or $\cos\theta = -1$.

$$\cos\theta = -\tfrac{1}{2} \implies \theta = 120^\circ,\ 240^\circ$$

$$\cos\theta = -1 \implies \theta = 180^\circ$$

$\theta = 120^\circ, \quad \theta = 180^\circ, \quad \theta = 240^\circ$

Question 8 (Jun 2014, Q7)

Worked Solution

$f(x) = 12 - 22x + 9x^2 - x^3$

Rewrite in descending powers: $f(x) = -x^3 + 9x^2 - 22x + 12$.

Part (i): Remainder when divided by $(x+2)$.

$f(-2) = -(-8) + 9(4) - 22(-2) + 12 = 8 + 36 + 44 + 12 = 100$

Remainder $= 100$

Part (ii): Show $(3-x)$ is a factor.

$f(3) = -27 + 81 - 66 + 12 = 0$ □

Part (iii): Express as a product of a linear and quadratic factor.

Divide $-x^3 + 9x^2 - 22x + 12$ by $(3-x)$, equivalently by $-(x-3)$.

Method 1 — Long division (dividing by $(x-3)$, then adjusting sign):

$$\begin{array}{r} x^2 - 6x + 4 \\ x - 3\,\big)\, x^3 - 9x^2 + 22x - 12 \\ x^3 - 3x^2 \\ \hline -6x^2 + 22x \\ -6x^2 + 18x \\ \hline 4x - 12 \\ 4x - 12 \\ \hline 0 \end{array}$$

So $x^3 - 9x^2 + 22x - 12 = (x-3)(x^2 - 6x + 4)$, hence:

$$f(x) = -(x-3)(x^2 - 6x + 4) = (3-x)(x^2 - 6x + 4)$$

Method 2 — Grid method:

Work with $-(x-3)$ as divisor; seek $(3-x)(x^2 + bx + c) = (-x+3)(x^2 + bx + c)$:

$\times$	x^2	bx	c
$-x$	$-x^3$	$-bx^2$	$-cx$
$+3$	$3x^2$	$3bx$	$3c$

Match to $-x^3 + 9x^2 - 22x + 12$:

- x^2: $3 - b = 9 \Rightarrow b = -6$
- x^0: $3c = 12 \Rightarrow c = 4$
- Check x: $-c + 3b = -4 - 18 = -22$ ✓

$f(x) = (3-x)(x^2 - 6x + 4)$

Part (iv): Solve $f(x) = 0$.

$(3 - x) = 0 \Rightarrow x = 3$.

Quadratic formula on $x^2 - 6x + 4 = 0$:

$$x = \frac{6 \pm \sqrt{36 - 16}}{2} = \frac{6 \pm \sqrt{20}}{2} = 3 \pm \sqrt{5}$$

$x = 3, \quad x = 3 + \sqrt{5}, \quad x = 3 - \sqrt{5}$

End of Worked Solutions