Question 1

Worked Solution

$f(x) = 2x^3 - 7x^2 + 4x + 4$

Part (a): Use the factor theorem to show $(x-2)$ is a factor.

Substitute $x = 2$:

$$f(2) = 2(8) - 7(4) + 4(2) + 4 = 16 - 28 + 8 + 4 = 0$$

Since $f(2) = 0$, by the factor theorem $(x-2)$ is a factor. □

Part (b): Factorise $f(x)$ completely.

Divide $2x^3 - 7x^2 + 4x + 4$ by $(x-2)$.

Method 1 — Long division:

$$\begin{array}{r} 2x^2 - 3x - 2 \\ x-2\,\overline{)\,2x^3 - 7x^2 + 4x + 4} \\ \underline{2x^3 - 4x^2} \\ -3x^2 + 4x \\ \underline{-3x^2 + 6x} \\ -2x + 4 \\ \underline{-2x + 4} \\ 0 \end{array}$$

Method 2 — Grid method:

We seek $(x-2)(2x^2 + bx + c)$ such that the product equals $2x^3 - 7x^2 + 4x + 4$.

$\times$	$2x^2$	bx	c
x	$2x^3$	bx^2	cx
-2	$-4x^2$	$-2bx$	$-2c$

Match coefficients:

- x^3: gives 2 ✓
- x^2: $b - 4 = -7 \Rightarrow b = -3$
- x^0: $-2c = 4 \Rightarrow c = -2$
- Check x: $c - 2b = -2 - 2(-3) = -2 + 6 = 4$ ✓

So the quotient is $2x^2 - 3x - 2$.

Factorise $2x^2 - 3x - 2 = (2x+1)(x-2)$.

$f(x) = (x-2)^2(2x+1)$

Question 2

Worked Solution

$f(x) = 3x^3 - 5x^2 - 16x + 12$

Part (a): Find the remainder when $f(x)$ is divided by $(x - 2)$.

Substitute $x = 2$:

$$f(2) = 3(8) - 5(4) - 16(2) + 12 = 24 - 20 - 32 + 12 = -16$$

Remainder $= -16$

Part (b): Given $(x + 2)$ is a factor, factorise $f(x)$ completely.

Divide $3x^3 - 5x^2 - 16x + 12$ by $(x + 2)$.

Method 1 — Long division:

$$\begin{array}{r} 3x^2 - 11x + 6 \\ x+2 \overline{\smash{\big)}\, 3x^3 - 5x^2 - 16x + 12} \\ \underline{3x^3 + 6x^2} \\ -11x^2 - 16x \\ \underline{-11x^2 - 22x} \\ 6x + 12 \\ \underline{6x + 12} \\ 0 \end{array}$$

Method 2 — Grid method:

Seek $(x + 2)(3x^2 + bx + c)$:

$\times$	$3x^2$	bx	c
x	$3x^3$	bx^2	cx
$+2$	$6x^2$	$2bx$	$2c$

Match coefficients:

- x^2: $b + 6 = -5 \Rightarrow b = -11$
- x^0: $2c = 12 \Rightarrow c = 6$
- Check x: $c + 2b = 6 + 2(-11) = 6 - 22 = -16$ ✓

Quotient is $3x^2 - 11x + 6$.

Factorise $3x^2 - 11x + 6 = (3x - 2)(x - 3)$.

$f(x) = (x + 2)(3x - 2)(x - 3)$

Question 3

Worked Solution

$f(x) = 4x^3 - 12x^2 + 2x - 6$

Part (a): Show $(x-3)$ is a factor.

Substitute $x = 3$:

$$f(3) = 4(27) - 12(9) + 2(3) - 6 = 108 - 108 + 6 - 6 = 0$$

Since $f(3) = 0$, by the factor theorem $(x-3)$ is a factor. □

Part (b): Show that $x = 3$ is the only real root of $f(x) = 0$.

Divide $4x^3 - 12x^2 + 2x - 6$ by $(x-3)$.

Method 1 — Long division:

$$\begin{array}{r}
4x^2 + 2 \\
x-3 \overline{\smash{)}\, 4x^3 - 12x^2 + 2x - 6} \\
\underline{4x^3 - 12x^2} \\
0x^2 + 2x - 6 \\
\underline{2x - 6} \\
0
\end{array}$$

Method 2 — Grid method:

Seek $(x-3)(4x^2 + bx + c)$:

$\times$	$4x^2$	bx	c
x	$4x^3$	bx^2	cx
-3	$-12x^2$	$-3bx$	$-3c$

Match coefficients:

- x^2: $b - 12 = -12 \Rightarrow b = 0$
- x^0: $-3c = -6 \Rightarrow c = 2$
- Check x: $c - 3b = 2 - 0 = 2$ ✓

So $f(x) = (x-3)(4x^2 + 2)$.

Now consider $4x^2 + 2 = 0$: this gives $x^2 = -\frac{1}{2}$, which has no real solutions (since $x^2 \geq 0$ for all real x).

Therefore $x = 3$ is the only real root. □

Question 4

Worked Solution

$f(x) = (x-4)(x^2-3x+k)-42$, where k is a constant.

Given $(x+2)$ is a factor, so $f(-2)=0$:

$$(-2-4)((-2)^2-3(-2)+k)-42=0$$

$$(-6)(4+6+k)-42=0$$

$$-6(10+k)=42$$

$$10+k=-7 \implies k=-17$$

$k=-17$

Question 5

Worked Solution

$f(x) = 3x^3 + 2ax^2 - 4x + 5a$

Given $(x + 3)$ is a factor, so $f(-3) = 0$:

$$3(-3)^3 + 2a(-3)^2 - 4(-3) + 5a = 0$$

$$3(-27) + 2a(9) + 12 + 5a = 0$$

$$-81 + 18a + 12 + 5a = 0$$

$$23a - 69 = 0 \implies a = 3$$

$a = 3$

Question 6

Worked Solution

Let $\mathrm{p}(x) = x^3 - 2x^2 - 4x + 8$.

Part (a)(i): Remainder when divided by $(x - 3)$.

$\mathrm{p}(3) = 27 - 18 - 12 + 8 = 5$

Remainder $= 5$

Part (a)(ii): Remainder when divided by $(x + 2)$.

$\mathrm{p}(-2) = (-8) - 2(4) - 4(-2) + 8 = -8 - 8 + 8 + 8 = 0$

The remainder is 0, so $(x + 2)$ is a factor.

Remainder $= 0$; $(x + 2)$ is a factor

Part (b): Find all solutions to $x^3 - 2x^2 - 4x + 8 = 0$.

Since $(x + 2)$ is a factor, divide by $(x + 2)$.

Method 1 — Long division:

$$\begin{array}{r} x^2 - 4x + 4 \\ x+2 \,\big)\, \overline{x^3 - 2x^2 - 4x + 8} \\ \underline{x^3 + 2x^2} \\ -4x^2 - 4x \\ \underline{-4x^2 - 8x} \\ 4x + 8 \\ \underline{4x + 8} \\ 0 \end{array}$$

Method 2 — Grid method:

Seek $(x + 2)(x^2 + bx + c)$:

$\times$	x^2	bx	c
x	x^3	bx^2	cx
$+2$	$2x^2$	$2bx$	$2c$

Match coefficients:

- x^2: $b + 2 = -2 \Rightarrow b = -4$
- x^0: $2c = 8 \Rightarrow c = 4$
- Check x: $c + 2b = 4 - 8 = -4$ ✓

So $\mathrm{p}(x) = (x + 2)(x^2 - 4x + 4) = (x + 2)(x - 2)^2$.

Setting $\mathrm{p}(x) = 0$:

$x = -2$ or $x = 2$ (repeated root)

Question 7

Worked Solution

$g(x) = 4x^3 - 12x^2 - 15x + 50$

Part (a): Show $(x + 2)$ is a factor.

Substitute $x = -2$:

$$g(-2) = 4(-8) - 12(4) - 15(-2) + 50 = -32 - 48 + 30 + 50 = 0$$

Since $g(-2) = 0$, $(x + 2)$ is a factor. □

Part (b): Show $g(x) = (x + 2)(ax + b)^2$ and find a, b.

Divide $4x^3 - 12x^2 - 15x + 50$ by $(x + 2)$.

Method 1 — Long division:

$$\begin{array}{r l}
 & 4x^2 - 20x + 25 \\
x + 2 \;\big) & 4x^3 - 12x^2 - 15x + 50 \\
 & 4x^3 + 8x^2 \\
 & -20x^2 - 15x \\
 & -20x^2 - 40x \\
 & 25x + 50 \\
 & 25x + 50 \\
 & 0
\end{array}$$

Method 2 — Grid method:

Seek $(x + 2)(4x^2 + bx + c)$:

$\times$	$4x^2$	bx	c
x	$4x^3$	bx^2	cx
$+2$	$8x^2$	$2bx$	$2c$

Match coefficients:

- x^2: $b + 8 = -12 \Rightarrow b = -20$
- x^0: $2c = 50 \Rightarrow c = 25$
- Check x: $c + 2b = 25 - 40 = -15$ ✓

So $g(x) = (x + 2)(4x^2 - 20x + 25)$.

Recognise $4x^2 - 20x + 25 = (2x - 5)^2$.

$$g(x) = (x + 2)(2x - 5)^2, \quad a = 2,\ b = -5$$

Part (c): From $g(x) = (x + 2)(2x - 5)^2$:

The roots are $x = -2$ (simple) and $x = \frac{5}{2}$ (repeated).

From the sketch, the curve crosses the x-axis at $x = -2$ (going from negative to positive) and touches at $x = \frac{5}{2}$.

(i) $g(x) \leq 0$**:**

The graph is below (or on) the x-axis for $x \leq -2$ and at the touching point $x = \frac{5}{2}$.

$$x \leq -2 \quad \text{or} \quad x = \frac{5}{2}$$

(ii) $g(2x) = 0$**:**

Replace x with $2x$ in the roots: $2x = -2 \Rightarrow x = -1$ and $2x = \frac{5}{2} \Rightarrow x = \frac{5}{4}$.

$$x = -1 \quad \text{or} \quad x = \frac{5}{4}$$

End of Worked Solutions