Question 1

Worked Solution

Given $f'(x) = \dfrac{(3 - x^2)^2}{x^2}$, $x \neq 0$.

(a) Expand the numerator:

$$(3 - x^2)^2 = 9 - 6x^2 + x^4$$

Divide each term by x^2:

$$f'(x) = \frac{9}{x^2} - 6 + \frac{x^4}{x^2} = 9x^{-2} - 6 + x^2$$

$A = -6,\ B = 1$

(b) Differentiate $f'(x) = 9x^{-2} - 6 + x^2$:

$$f''(x) = -18x^{-3} + 2x$$

$f''(x) = -18x^{-3} + 2x$

(c) Integrate $f'(x) = 9x^{-2} - 6 + x^2$:

$$f(x) = \frac{9x^{-1}}{-1} - 6x + \frac{x^3}{3} + c = -9x^{-1} - 6x + \frac{x^3}{3} + c$$

Use the point $(-3, 10)$:

$$10 = \frac{-9}{-3} - 6(-3) + \frac{(-3)^3}{3} + c = 3 + 18 - 9 + c = 12 + c$$

$$c = -2$$

$f(x) = -9x^{-1} - 6x + \dfrac{x^3}{3} - 2$

Question 2

Worked Solution

$\frac{\mathrm{d}y}{\mathrm{d}x} = -x^3 + \frac{4x-5}{2x^3}$, $x \neq 0$.

Rewrite by splitting the fraction:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -x^3 + \frac{4x}{2x^3} - \frac{5}{2x^3} = -x^3 + 2x^{-2} - \frac{5}{2}x^{-3}$$

Integrate term by term:

$$y = -\frac{x^4}{4} + \frac{2x^{-1}}{-1} - \frac{5}{2} \cdot \frac{x^{-2}}{-2} + c = -\frac{1}{4}x^4 - 2x^{-1} + \frac{5}{4}x^{-2} + c$$

Use $y = 7$ at $x = 1$:

$$7 = -\frac{1}{4} - 2 + \frac{5}{4} + c = -\frac{1}{4} - 2 + \frac{5}{4} + c = 1 - 2 + c = -1 + c$$

$$c = 8$$

$$y = -\frac{1}{4}x^4 - 2x^{-1} + \frac{5}{4}x^{-2} + 8$$

Question 3

Worked Solution

$\frac{dy}{dx} = \frac{(x^2+3)^2}{x^2}$, $x \neq 0$.

(a) Expand the numerator:

$$(x^2+3)^2 = x^4 + 6x^2 + 9$$

Divide by x^2:

$$\frac{dy}{dx} = \frac{x^4 + 6x^2 + 9}{x^2} = x^2 + 6 + 9x^{-2} \qquad \checkmark$$

(b) Integrate:

$$y = \frac{x^3}{3} + 6x + \frac{9x^{-1}}{-1} + c = \frac{x^3}{3} + 6x - 9x^{-1} + c$$

Use point $(3, 20)$:

$$20 = \frac{27}{3} + 18 - \frac{9}{3} + c = 9 + 18 - 3 + c = 24 + c$$

$$c = -4$$

$$y = \frac{x^3}{3} + 6x - 9x^{-1} - 4$$

Question 4

Worked Solution

$\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^{-\frac{1}{2}} + x\sqrt{x}$, $x > 0$.

Note that $x\sqrt{x} = x^{\frac{3}{2}}$.

Integrate:

$$y = \frac{6x^{\frac{1}{2}}}{\frac{1}{2}} + \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + c = 12x^{\frac{1}{2}} + \frac{2}{5}x^{\frac{5}{2}} + c$$

Use $y = 37$ at $x = 4$:

$$37 = 12\sqrt{4} + \frac{2}{5}(\sqrt{4})^5 + c = 12(2) + \frac{2}{5}(32) + c = 24 + \frac{64}{5} + c$$

$$37 = 24 + 12.8 + c \implies c = 37 - 36.8 = \frac{1}{5}$$

$$y = 12x^{\frac{1}{2}} + \frac{2}{5}x^{\frac{5}{2}} + \frac{1}{5}$$

Question 5

Worked Solution

$\frac{dy}{dx} = \frac{6x + 3x^{\frac{5}{2}}}{\sqrt{x}}$, $y = 90$ when $x = 4$.

(a) Divide each term by $x^{\frac{1}{2}}$:

$$\frac{6x}{\sqrt{x}} + \frac{3x^{\frac{5}{2}}}{\sqrt{x}} = 6x^{\frac{1}{2}} + 3x^2 = 6x^p + 3x^q$$

$p = \frac{1}{2}, \quad q = 2$

(b) Integrate:

$$y = \frac{6x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{3x^3}{3} + c = 4x^{\frac{3}{2}} + x^3 + c$$

Use $y = 90$ at $x = 4$:

$$90 = 4(4)^{\frac{3}{2}} + (4)^3 + c = 4(8) + 64 + c = 32 + 64 + c = 96 + c$$

$$c = -6$$

$y = 4x^{\frac{3}{2}} + x^3 - 6$

Question 6

Worked Solution

$f'(x) = 12x^2 - 8x + 1$, curve passes through $(-1, 0)$.

Integrate:

$$f(x) = \frac{12x^3}{3} - \frac{8x^2}{2} + x + c = 4x^3 - 4x^2 + x + c$$

Use $f(-1) = 0$:

$$0 = 4(-1)^3 - 4(-1)^2 + (-1) + c = -4 - 4 - 1 + c = -9 + c$$

$$c = 9$$

$f(x) = 4x^3 - 4x^2 + x + 9$

Question 7

Worked Solution

$f'(x) = 3x^2 - 3x + 5$, curve passes through $(2, 10)$.

Integrate:

$$f(x) = \frac{3x^3}{3} - \frac{3x^2}{2} + 5x + c = x^3 - \frac{3}{2}x^2 + 5x + c$$

Use $f(2) = 10$:

$$10 = 8 - \frac{3}{2}(4) + 10 + c = 8 - 6 + 10 + c = 12 + c$$

$$c = -2$$

So $f(x) = x^3 - \frac{3}{2}x^2 + 5x - 2$.

Find $f(1)$:

$$f(1) = 1 - \frac{3}{2} + 5 - 2 = \frac{2 - 3 + 10 - 4}{2} = \frac{5}{2}$$

$f(1) = \frac{5}{2}$

Question 8

Worked Solution

$\frac{dy}{dx} = 3x - \frac{5}{\sqrt{x}} - 2 = 3x - 5x^{-\frac{1}{2}} - 2$, $x > 0$, point $P(4, 5)$.

(a) Integrate:

$$y = \frac{3x^2}{2} - \frac{5x^{\frac{1}{2}}}{\frac{1}{2}} - 2x + c = \frac{3}{2}x^2 - 10x^{\frac{1}{2}} - 2x + c$$

Use $y = 5$ at $x = 4$:

$$5 = \frac{3}{2}(16) - 10(2) - 2(4) + c = 24 - 20 - 8 + c = -4 + c$$

$$c = 9$$

$$f(x) = \frac{3}{2}x^2 - 10x^{\frac{1}{2}} - 2x + 9$$

(b) Find the gradient at $P(4, 5)$:

$$m = 3(4) - \frac{5}{\sqrt{4}} - 2 = 12 - \frac{5}{2} - 2 = \frac{15}{2}$$

Equation of tangent through $(4, 5)$:

$$y - 5 = \frac{15}{2}(x - 4) \implies 2y - 10 = 15x - 60 \implies 2y - 15x + 50 = 0$$

$$2y - 15x + 50 = 0$$

End of Worked Solutions