Question 1 (OCR 4723, Jan 2006, Q6)

Worked Solution

Part (a): X is increasing exponentially. Given $X = 275$ at $t = 0$ and $X = 440$ at $t = 10$, find X at $t = 20$.

The ratio $\frac{440}{275} = \frac{8}{5}$ is the growth factor over 10 units of time. Over another 10 units it multiplies by the same ratio:

$$X(20) = 440 \times \frac{440}{275} = 440 \times \frac{8}{5} = 704.$$

$X = 704$

Part (b)(i): $Y = 80e^{-0.02t}$. Find t when $Y = 20$, correct to 2 s.f.

$$80e^{-0.02t} = 20 \implies e^{-0.02t} = \frac{1}{4} \implies -0.02t = \ln\left(\tfrac{1}{4}\right) = -\ln 4.$$

$$t = \frac{\ln 4}{0.02} \approx 69.$$

$t \approx 69$ (to 2 s.f.)

Part (b)(ii): Rate at which Y is decreasing at $t = 30$.

Differentiate: $\frac{\mathrm{d}Y}{\mathrm{d}t} = -0.02 \times 80e^{-0.02t} = -1.6e^{-0.02t}$.

At $t = 30$:

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = -1.6e^{-0.6} \approx -0.879.$$

Rate of decrease ≈ 0.88 (to 2 s.f.).

Rate of decrease ≈ 0.88 per unit time (to 2 s.f.)

Question 2 (OCR 4723, Jun 2008, Q7)

Worked Solution

Plants double every 9 years, currently 42 plants. $N = A \times 2^{kt}$ and $N = Ae^{mt}$.

Part (i): Find A, k, m.

At $t = 0$: $N = 42$, so $A = 42$.

The doubling time is 9 years, so $2^{9k} = 2 \Rightarrow 9k = 1 \Rightarrow k = \frac{1}{9}$.

For $N = 42e^{mt}$: equate with $42 \times 2^{t/9}$, so $e^m = 2^{1/9}$, giving $m = \frac{\ln 2}{9}$.

$$A = 42, \quad k = \frac{1}{9}, \quad m = \frac{\ln 2}{9} \approx 0.077$$

Part (ii): Find t when $N = 100$.

$$42e^{mt} = 100 \implies e^{mt} = \frac{100}{42} \implies mt = \ln\left(\tfrac{100}{42}\right) \implies t = \frac{9\ln(100/42)}{\ln 2} \approx 11.3.$$

$t \approx 11.3$ years (to 3 s.f.)

Part (iii): Rate of increase at $t = 35$.

$\frac{dN}{dt} = 42m\,e^{mt}$. At $t = 35$:

$$\frac{dN}{dt} = 42 \times \frac{\ln 2}{9} \times e^{35\ln 2/9} \approx 3.235 \times e^{0.077 \times 35} \approx 3.235 \times 14.80 \approx 47.9.$$

Rate ≈ 47.9 plants per year

Question 3 (OCR 4723, Jan 2009, Q5)

Worked Solution

$M = 40e^{kt}$; given $M = 80$ at $t = 21$.

Part (i)(a): Missing table values.

At $t = 0$: $M = 40$. Find k from $t = 21$, $M = 80$:

$$40e^{21k} = 80 \implies e^{21k} = 2 \implies 21k = \ln 2 \implies k = \frac{\ln 2}{21} \approx 0.033.$$

At $t = 63$: $M = 40e^{63k} = 40\left(e^{21k}\right)^3 = 40 \times 2^3 = 320.$

At $t = 0$: $M = 40$. At $t = 63$: $M = 320$.

Part (i)(b):

$k = \frac{\ln 2}{21} \approx 0.033$

Part (ii): Rate of increase at $t = 21$.

$\frac{dM}{dt} = 40k\, e^{kt}$. At $t = 21$:

$$\frac{dM}{dt} = 40 \times \frac{\ln 2}{21} \times e^{21k} = 40 \times \frac{\ln 2}{21} \times 2 \approx 2.64.$$

Rate ≈ 2.64 grams/hour

Question 4 (OCR 4723, Jan 2012, Q7)

Worked Solution

$M = 40e^{-0.132t}$ (substance A).

Part (i)(a): Time for mass to decrease to 25% of initial value.

25% of 40 = 10:

$$40e^{-0.132t} = 10 \implies e^{-0.132t} = 0.25 \implies -0.132t = \ln(0.25) \implies t = \frac{\ln 0.25}{-0.132} \approx 10.5.$$

$t \approx 10.5$ years

Part (i)(b): Rate of decrease at $t = 5$.

$\frac{\mathrm{d}M}{\mathrm{d}t} = -0.132 \times 40\,e^{-0.132t} = -5.28e^{-0.132t}$.

At $t = 5$:

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -5.28e^{-0.66} \approx -2.73.$$

Rate of decrease ≈ 2.73 g/year

Part (ii): Substance B: 40 g initially, 31.4 g after 2 years. Find mass after a further year (i.e. at $t = 3$).

The model is $M_B = 40e^{kt}$. Use $t = 2$, $M_B = 31.4$:

$$40e^{2k} = 31.4 \implies e^{2k} = \frac{31.4}{40} \implies 2k = \ln\left(\tfrac{31.4}{40}\right) \implies k \approx -0.1205.$$

At $t = 3$:

$$M_B = 40e^{3k} = 40e^{3\times(-0.1205)} \approx 40e^{-0.3616} \approx 27.8 \text{ g}.$$

Mass of substance B at $t = 3$ years ≈ 27.8 g

Question 5 (OCR 4723, Jun 2014, Q5)

Worked Solution

Part (a): $M = 58e^{-0.33t}$. Rate of decrease at $t = 4$.

$\frac{\mathrm{d}M}{\mathrm{d}t} = -0.33 \times 58\,e^{-0.33t} = -19.14e^{-0.33t}$.
At $t = 4$:

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -19.14e^{-1.32} \approx -19.14 \times 0.2671 \approx -5.11.$$

Rate of decrease ≈ 5.1 g/year (to 2 s.f.)

Part (b): Second substance: $M = 42.0$ g at $t = 0$, $M = 51.8$ g at $t = 6$. Find mass at $t = 24$.

The model is $M = 42e^{kt}$. Use $t = 6$, $M = 51.8$:

$$42e^{6k} = 51.8 \implies e^{6k} = \frac{51.8}{42} \implies 6k = \ln\left(\tfrac{51.8}{42}\right) \implies k = \frac{\ln(51.8/42)}{6} \approx 0.035.$$

At $t = 24$:

$$M = 42e^{24k} = 42\left(e^{6k}\right)^4 = 42 \times \left(\frac{51.8}{42}\right)^4 \approx 42 \times 2.319 \approx 97.4 \text{ g}.$$

Mass at $t = 24$ years ≈ 97.2 g (between 97.1 and 97.3)

Question 6 (OCR 4723, Jun 2016, Q3)

Worked Solution

Mass m decreasing exponentially: $m = 200$ at $t = 0$, $m = 160$ at $t = 5$.

Part (i): Missing values: $t = 10$ and $t = 25$.

Ratio per 5 years: $\frac{160}{200} = 0.8$.

At $t = 10$: $m = 200 \times 0.8^2 = 200 \times 0.64 = 128$.

At $t = 25$: $m = 200 \times 0.8^5 = 200 \times 0.32768 \approx 65.5$.

$m(10) = 128, \quad m(25) \approx 65.5$

Part (ii): Find t (nearest integer) when $m = 50$.

The model is $m = 200e^{kt}$. From $t = 5$, $m = 160$:

$$e^{5k} = 0.8 \implies k = \frac{\ln 0.8}{5}.$$

Set $200e^{kt} = 50$:

$$e^{kt} = 0.25 \implies kt = \ln(0.25) \implies t = \frac{\ln 0.25}{k} = \frac{5 \ln 0.25}{\ln 0.8}.$$

$$t = \frac{5 \times (-1.3863)}{-0.22314} \approx 31.07 \approx 31.$$

$t = 31$ years (to nearest integer)

End of Worked Solutions