Question 1

Worked Solution

$V = 15700e^{-0.25t} + 2300$, $t \geq 0$.

Part (a): Initial value (at $t = 0$).

$V = 15700e^{0} + 2300 = 15700 + 2300$.

Initial value = £18,000

Part (b)(i): Show that $3925e^{-0.25T} = 500$.

The rate of decrease of V is $\frac{\mathrm{d}V}{\mathrm{d}t}$. Differentiate:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 15700 \times (-0.25)e^{-0.25t} = -3925e^{-0.25t}.$$

At $t = T$ the rate is -500 (decreasing at £500/year):

$$-3925e^{-0.25T} = -500 \implies 3925e^{-0.25T} = 500. \quad \square$$

Part (b)(ii): Solve for T.

$$e^{-0.25T} = \frac{500}{3925} = \frac{20}{157} \implies -0.25T = \ln\left(\frac{20}{157}\right) \implies T = \frac{\ln(20/157)}{-0.25}.$$

$$T \approx 8.24 \text{ years.}$$

Convert 0.24 years to months: $0.24 \times 12 \approx 2.88 \approx 3$ months.

$T \approx 8$ years 3 months

Part (c): Value of A.

As $t \to \infty$, $e^{-0.25t} \to 0$, so $V \to 2300$.

$A = 2300$

Part (d): Limitation of the model.

E.g. Other factors affect car value (condition, mileage, accidents); £2300 is unrealistically high as a scrap value.

Question 2

Worked Solution

$\theta = 120 - 100e^{-\lambda t}$, $0 \le t \le T$.

Part (a): Value of θ when $t = 0$.

$\theta = 120 - 100e^{0} = 120 - 100 = 20$.

$\theta = 20$

Part (b): Find exact λ given $\theta = 70$ when $t = 40$.

Substitute:

$$70 = 120 - 100e^{-40\lambda} \implies 100e^{-40\lambda} = 50 \implies e^{-40\lambda} = \frac{1}{2}.$$

Take natural logs:

$$-40\lambda = \ln\left(\tfrac{1}{2}\right) = -\ln 2 \implies \lambda = \frac{\ln 2}{40}.$$

$\lambda = \dfrac{\ln 2}{40}$

Part (c): Find T when $\theta = 100$.

$$100 = 120 - 100e^{-\lambda T} \implies 100e^{-\lambda T} = 20 \implies e^{-\lambda T} = 0.2.$$

$$-\lambda T = \ln(0.2) \implies T = \frac{\ln(0.2)}{-\lambda} = \frac{\ln 5}{\lambda} = \frac{40\ln 5}{\ln 2} \approx 93.$$

$T \approx 93$ seconds

Question 3

Worked Solution

$P = \dfrac{800e^{0.1t}}{1+3e^{0.1t}}$, $t \geq 0$.

Part (a): Population at $t = 0$.

$$P = \frac{800e^0}{1+3e^0} = \frac{800}{1+3} = \frac{800}{4} = 200.$$

200 primroses

Part (b): Find exact t when $P = 250$.

$$250 = \frac{800e^{0.1t}}{1+3e^{0.1t}}.$$

Cross-multiply:

$$250(1+3e^{0.1t}) = 800e^{0.1t} \implies 250+750e^{0.1t} = 800e^{0.1t} \implies 250 = 50e^{0.1t} \implies e^{0.1t} = 5.$$

Take natural logs:

$$0.1t = \ln 5 \implies t = \frac{\ln 5}{0.1} = 10\ln 5.$$

$t = 10\ln 5$

Part (c): Why can P never equal 270?

Rewrite P by dividing numerator and denominator by $e^{0.1t}$:

$$P = \frac{800}{e^{-0.1t}+3}.$$

As $t \to \infty$, $e^{-0.1t} \to 0$, so $P \to \dfrac{800}{3} \approx 266.7$. Since P increases towards but never reaches $\dfrac{800}{3} < 270$, the population can never be 270.

As $t \to \infty$, $P \to \frac{800}{3} \approx 266.7$, which is less than 270, so P can never reach 270.

Question 4

Worked Solution

$x = De^{-0.2t}$ (amount of antibiotic in bloodstream).

Part (a): Amount 4 hours after first dose of 15 mg.

$$x = 15e^{-0.2\times 4} = 15e^{-0.8} \approx 6.740 \text{ mg}.$$

$x \approx 6.740$ mg

Part (b): Show total amount 2 hours after second dose is 13.754 mg.

The second dose is given 5 hours after the first. So 2 hours after the second dose:

- First dose has been in the body for $5 + 2 = 7$ hours: $15e^{-0.2\times 7} = 15e^{-1.4}$.
- Second dose has been in the body for 2 hours: $15e^{-0.2\times 2} = 15e^{-0.4}$.

Total:

$$15e^{-1.4} + 15e^{-0.4} \approx 3.704 + 10.050 = 13.754 \text{ mg}. \quad \square$$

Part (c): Show $T = 5\ln\left(2 + \frac{2}{e}\right)$.

At time T hours after the second dose, the first dose has been in the body for $T + 5$ hours:

$$15e^{-0.2T} + 15e^{-0.2(T+5)} = 7.5.$$

Factor out $15e^{-0.2T}$, noting $e^{-0.2(T+5)} = e^{-0.2T} \cdot e^{-1}$:

$$15e^{-0.2T}(1 + e^{-1}) = 7.5 \implies e^{-0.2T} = \frac{7.5}{15(1 + e^{-1})} = \frac{1}{2(1 + e^{-1})}.$$

Take natural logs:

$$-0.2T = \ln\left(\frac{1}{2(1 + e^{-1})}\right) = -\ln\left(2(1 + e^{-1})\right).$$

$$T = 5\ln\left(2(1 + e^{-1})\right) = 5\ln\left(2 + \frac{2}{e}\right). \quad \square$$

$T = 5\ln\left(2 + \frac{2}{e}\right)$, where $a = 5$, $b = 2$.

Question 5

Worked Solution

$R = 1000e^{-ct}$, $t \geq 0$.

Part (a): Atoms at start ($t = 0$).

$R = 1000e^{0} = 1000$.

1000 atoms

Part (b): Find c given the half-life is 5730 years.

When half the substance has decayed, $R = 500$:

$$1000e^{-5730c} = 500 \implies e^{-5730c} = \frac{1}{2} \implies -5730c = \ln\left(\tfrac{1}{2}\right) = -\ln 2.$$

$$c = \frac{\ln 2}{5730} \approx 0.000121.$$

$c = \dfrac{\ln 2}{5730} \approx 0.000121$

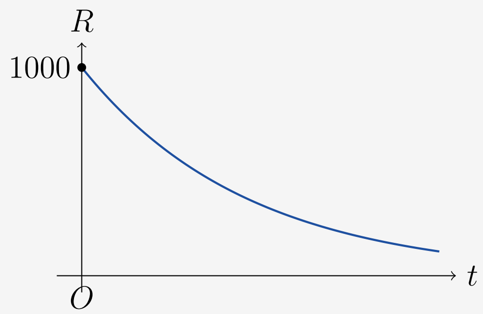
Part (c): R when $t = 22920$.

Note $22920 = 4 \times 5730$ (four half-lives):

$$R = 1000e^{-c\times 22920} = 1000 \times \left(e^{-5730c}\right)^4 = 1000 \times \left(\tfrac{1}{2}\right)^4 = 1000 \times \tfrac{1}{16} = 62.5.$$

$R = 62.5$ atoms

Part (d): Sketch of R against t.



Starts at $R = 1000$ when $t = 0$; decreasing curve approaching the t-axis asymptotically.

Question 6

Worked Solution

$m = pe^{-kt}$, where $m = 7.5$ at $t = 0$ and $m = 2.5$ at $t = 4$.

Part (a): Value of p.

At $t = 0$: $m = pe^0 = p = 7.5$.

$p = 7.5$

Part (b): Show $k = \frac{1}{4}\ln 3$.

At $t = 4$, $m = 2.5$:

$$2.5 = 7.5e^{-4k} \implies e^{-4k} = \frac{2.5}{7.5} = \frac{1}{3} \implies -4k = \ln\left(\tfrac{1}{3}\right) = -\ln 3.$$

$$k = \frac{\ln 3}{4} = \frac{1}{4}\ln 3. \quad \square$$

Part (c): Find t when $\dfrac{\mathrm{d}m}{\mathrm{d}t} = -0.6\ln 3$.

Differentiate $m = 7.5e^{-kt}$:

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -k \cdot 7.5e^{-kt} = -\frac{\ln 3}{4} \times 7.5e^{-\frac{\ln 3}{4}t}.$$

Set equal to $-0.6\ln 3$:

$$-\frac{7.5\ln 3}{4}e^{-\frac{\ln 3}{4}t} = -0.6\ln 3 \implies \frac{7.5}{4}e^{-\frac{\ln 3}{4}t} = 0.6.$$

$$e^{-\frac{\ln 3}{4}t} = \frac{0.6 \times 4}{7.5} = \frac{2.4}{7.5} = 0.32.$$

Take natural logs:

$$-\frac{\ln 3}{4}t = \ln(0.32) \implies t = \frac{-4\ln(0.32)}{\ln 3} \approx \frac{4 \times 1.139}{1.099} \approx 4.15.$$

$t \approx 4.15$ days

Question 7

Worked Solution

This question uses the same model as Q1: $V = 15700e^{-0.25t} + 2300$.

Part (a):

Initial value = £18,000

Part (b)(i): Show $3925e^{-0.25T} = 500$.

Differentiate: $\dfrac{\mathrm{d}V}{\mathrm{d}t} = -3925e^{-0.25t}$. Setting rate = −500:

$$-3925e^{-0.25T} = -500 \implies 3925e^{-0.25T} = 500. \quad \square$$

Part (b)(ii):

$$e^{-0.25T} = \frac{500}{3925} \implies -0.25T = \ln\left(\frac{20}{157}\right) \implies T \approx 8.24 \text{ years.}$$

$0.24 \times 12 \approx 3$ months.

$T \approx 8$ years 3 months

Part (c):

$A = 2300$

Part (d):

E.g. Other factors affect price (condition, mileage); £2300 is unrealistically high as a scrap floor value.

End of Worked Solutions