Question 1 (Jan 2007, Q1)

Worked Solution

Differentiate $6x^{5/2} + 4$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6 \times \tfrac{5}{2}\, x^{3/2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 15x^{3/2}$$

Question 2 (Jan 2009, Q7)

Worked Solution

$y = 4x^2 + \frac{1}{x} = 4x^2 + x^{-1}$

$$\frac{dy}{dx} = 8x - x^{-2} = 8x - \frac{1}{x^2}$$

Set $\frac{dy}{dx} = 0$:

$$8x - \frac{1}{x^2} = 0 \implies 8x^3 = 1 \implies x^3 = \frac{1}{8}$$

$x = \frac{1}{2}$

Question 3 (Jun 2007, Q9)

Worked Solution

$y = 2x^3 - 9x^2 + 12x - 2$

(i) $\frac{dy}{dx} = 6x^2 - 18x + 12$

At $x = 3$: gradient $= 54 - 54 + 12 = 12$.

When $x = 3$: $y = 54 - 81 + 36 - 2 = 7$.

Tangent at $(3, 7)$: $y - 7 = 12(x - 3)$, i.e. $y = 12x - 29$.

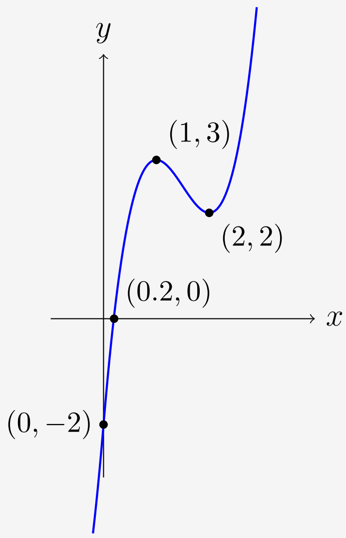
Check $(-1, -41)$: $12(-1) - 29 = -41$. ✓

(ii) Set $\frac{dy}{dx} = 0$: $6x^2 - 18x + 12 = 0 \Rightarrow x^2 - 3x + 2 = 0 \Rightarrow (x-1)(x-2) = 0$.

$x = 1$: $y = 2 - 9 + 12 - 2 = 3$. $x = 2$: $y = 16 - 36 + 24 - 2 = 2$.

Turning points: $(1, 3)$ and $(2, 2)$.

(iii) Cubic with positive leading coefficient, only real root at $x = 0.2$, local max at $(1, 3)$, local min at $(2, 2)$, y-intercept at $(0, -2)$.



Question 4 (Jun 2009, Q6)

Worked Solution

$y = x^3 - 6x^2 - 15x, \quad \frac{dy}{dx} = 3x^2 - 12x - 15$

Set $\frac{dy}{dx} = 0$: $3x^2 - 12x - 15 = 0 \Rightarrow x^2 - 4x - 5 = 0 \Rightarrow (x-5)(x+1) = 0$

Turning points at $x = 5$ and $x = -1$.

For increasing function: $\frac{dy}{dx} > 0$, i.e. $3(x-5)(x+1) > 0$.

$x < -1$ or $x > 5$

Question 5 (Jan 2013, Q6)

Worked Solution

$f(x) = 2x^3 + 9x^2 - 24x$

$f'(x) = 6x^2 + 18x - 24$

Set $f'(x) = 0$: $6x^2 + 18x - 24 = 0 \Rightarrow x^2 + 3x - 4 = 0 \Rightarrow (x + 4)(x - 1) = 0$

So $x = -4$ or $x = 1$.

For $f(x)$ increasing: $f'(x) > 0$, i.e. $6(x + 4)(x - 1) > 0$.

$x < -4$ or $x > 1$

Question 6 (Jun 2009, Q12)

Worked Solution

(i) At $x = 3$: $y = 9 - 7 = 2$. At $x = 3.1$: $y = 9.61 - 7 = 2.61$.

$$\text{Gradient} = \frac{2.61 - 2}{3.1 - 3} = \frac{0.61}{0.1}$$

$= 6.1$

(ii) $\dfrac{f(3+h) - f(3)}{h} = \dfrac{(3+h)^2 - 7 - (9-7)}{h} = \dfrac{6h + h^2}{h}$

$= 6 + h$

(iii) As $h \to 0$, $6 + h \to 6$.

Gradient at $x = 3$ is 6.

(iv) At $x = 3$, $y = 2$, gradient $= 6$: $y - 2 = 6(x - 3)$

$y = 6x - 16$

(v) P: set $y = 0$ in tangent: $x = \frac{16}{6} = \frac{8}{3}$. Q: $x^2 = 7$, $x = \sqrt{7}$ (positive).

$PQ = \sqrt{7} - \frac{8}{3} \approx 2.6458 - 2.6\overline{6}$

$PQ \approx 0.021$

Question 7 (Jan 2006, Q11)

Worked Solution

$y = x^3 - 6x + 2$

(i) $\dfrac{dy}{dx} = 3x^2 - 6$

(ii) Decreasing when $\dfrac{dy}{dx} < 0$: $3x^2 - 6 < 0 \Rightarrow x^2 < 2$

$-\sqrt{2} < x < \sqrt{2}$

(iii) At $(-1, 7)$: gradient $= 3(1) - 6 = -3$.

Tangent: $y - 7 = -3(x + 1) \Rightarrow y = -3x + 4$.

Find where tangent meets curve again: $x^3 - 6x + 2 = -3x + 4$

$$x^3 - 3x - 2 = 0 \implies (x + 1)^2(x - 2) = 0$$

So second intersection at $x = 2$: $y = -3(2) + 4 = -2$.

Tangent meets curve again at $(2, -2)$.

Question 8 (Jan 2007, Q5)

Worked Solution

$y = \dfrac{4}{x^2} = 4x^{-2}$

(i) At $A(2, 1)$, B has $x = 2.1$: $y_B = \dfrac{4}{4.41} \approx 0.9070$.

Gradient of $AB = \dfrac{0.9070 - 1}{0.1}$

≈ -0.93

(ii) Any x strictly between 1.91 and 2, or strictly between 2 and 2.1 (not 2.1).

(iii) $\dfrac{dy}{dx} = -8x^{-3}$. At $x = 2$: gradient $= -\dfrac{8}{8}$

$= -1$

Question 9 (Jun 2014, Q11)

Worked Solution

$y = x - \dfrac{4}{x^2} = x - 4x^{-2}$

(i) $\dfrac{dy}{dx} = 1 + 8x^{-3} = 1 + \dfrac{8}{x^3}, \quad \dfrac{d^2y}{dx^2} = -24x^{-4} = -\dfrac{24}{x^4}$

(ii) Set $\dfrac{dy}{dx} = 0$: $1 + \dfrac{8}{x^3} = 0 \Rightarrow x^3 = -8 \Rightarrow x = -2$.

$y = -2 - \dfrac{4}{4} = -3$. Stationary point $(-2, -3)$.

Check nature: $\dfrac{d^2y}{dx^2} = -\dfrac{24}{16} = -\dfrac{3}{2} < 0$, so maximum.

Maximum at $(-2, -3)$.

(iii) At $x = -1$: $y = -1 - 4 = -5$. Gradient $= 1 + \dfrac{8}{-1} = -7$. Normal gradient $= \dfrac{1}{7}$.

Normal: $y + 5 = \dfrac{1}{7}(x + 1)$, i.e. $7y + 35 = x + 1$

$-x + 7y + 34 = 0$

Question 10 (Jun 2016, Q10)

Worked Solution

(i) At $x = 5$: $y = 15$. At $x = 5.1$: $y = 15.81$. Gradient $= \dfrac{0.81}{0.1}$

$= 8.1$

(ii) $\dfrac{f(5+h) - f(5)}{h} = \dfrac{(5+h)^2 - 2(5+h) - 15}{h} = \dfrac{8h + h^2}{h}$

$= 8 + h$

(iii) As $h \to 0$, $8 + h \to 8$.

Gradient at $x = 5$ is 8.

(iv) $y = 8x - 25$. x-intercept: $x = \frac{25}{8}$. y-intercept: $y = -25$.

Area $= \frac{1}{2} \times \frac{25}{8} \times 25 = \dfrac{625}{16}$

Area $= \dfrac{625}{16} = 39.0625$

Question 11 (OCR 4721, Jun 2015, Q9)

Worked Solution

$y = 2x^3 - ax^2 + 8x + 2$

(i) $\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 - 2ax + 8$. At $x = 4$ (stationary): $6(16) - 8a + 8 = 0 \Rightarrow 104 - 8a = 0$

$a = 13$

(ii) $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12x - 26$. At $x = 4$: $48 - 26 = 22 > 0$

Minimum point.

(iii) Set $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ with $a = 13$: $6x^2-26x+8 = 0 \Rightarrow 3x^2-13x+4 = 0 \Rightarrow (3x-1)(x-4) = 0$

Other stationary point at $x = \frac{1}{3}$.

Question 12 (OCR 4721, Jun 2016, Q8)

Worked Solution

$y = 2x^2$, points $A(5, 50)$ and $B(5 + h, 2(5 + h)^2)$.

(i) $y_1 = 50$, $y_2 = 2(25 + 10h + h^2) = 50 + 20h + 2h^2$.

Gradient of $AB = \dfrac{20h + 2h^2}{h}$

$= 20 + 2h$ ✓

(ii) As $h \to 0$, $20 + 2h \to 20$. The gradient of the curve at A is 20.

(iii) Normal gradient at $A = -\dfrac{1}{20}$. Through $(5, 50)$: $y - 50 = -\dfrac{1}{20}(x - 5)$.

At $x = 0$: $y = 50 + \dfrac{5}{20} = 50\frac{1}{4}$.

y-coordinate of $C = 50\frac{1}{4}$

Question 13 (OCR 4721, Jun 2016, Q11)

Worked Solution

$y = 4x^2 + \frac{a}{x} + 5 = 4x^2 + ax^{-1} + 5$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 8x - ax^{-2}$

At stationary point: $8x - \frac{a}{x^2} = 0 \Rightarrow a = 8x^3$.

y-coordinate is 32: $4x^2 + 8x^3 \cdot x^{-1} + 5 = 32 \Rightarrow 4x^2 + 8x^2 + 5 = 32 \Rightarrow 12x^2 = 27 \Rightarrow x^2 = \frac{9}{4} \Rightarrow x = \frac{3}{2}$ (positive).

$a = 8\left(\frac{3}{2}\right)^3 = 8 \times \frac{27}{8}$

$a = 27$

Question 14 (OCR 4721, Jun 2017, Q11)

Worked Solution

$y = \dfrac{k}{x^2} = kx^{-2}$

(i) Gradient of given line $\frac{1}{2}y = 2 + 3x$ is 6. Normal gradient $= -\dfrac{1}{6}$.

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = -2kx^{-3}$. At $x = -3$: $-2k(-3)^{-3} = -2k \times (-\frac{1}{27}) = \dfrac{2k}{27}$.

Set equal to normal gradient: $\dfrac{2k}{27} = -\dfrac{1}{6} \Rightarrow k = -\dfrac{27}{12}$

$k = -\dfrac{9}{4}$

(ii) At $x = -3$: $y = -\dfrac{9}{4} \times \dfrac{1}{9} = -\dfrac{1}{4}$.

Normal through $(-3, -\frac{1}{4})$ with gradient 6: $y + \frac{1}{4} = 6(x + 3)$, i.e. $y = 6x + \frac{71}{4}$.

Multiply through by 4: $4y = 24x + 71$

$24x - 4y + 71 = 0$

Question 15 (Jun 2010, Q10)

Worked Solution

(i) $\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^3$. At $x = 2$: gradient $= 32$, $y = 16$. Tangent: $y - 16 = 32(x - 2)$

$y = 32x - 48$

(ii) $\frac{2.1^4 - 16}{0.1} = \frac{19.4481 - 16}{0.1}$

$= 34.481$

(iii)(A) $(2 + h)^4 = 16 + 4(8)h + 6(4)h^2 + 4(2)h^3 + h^4$

$= 16 + 32h + 24h^2 + 8h^3 + h^4$

(iii)(B) $\frac{(2 + h)^4 - 16}{h} = \frac{32h + 24h^2 + 8h^3 + h^4}{h}$

$= 32 + 24h + 8h^2 + h^3$

(iii)(C) As $h \to 0$, $32 + 24h + 8h^2 + h^3 \to 32$. The gradient of the tangent is the limit of the gradient of the chord, so the gradient of $y = x^4$ at $x = 2$ is **32**.

Question 16 (OCR H230/02, Sample Q7)

Worked Solution

Differentiate $f(x) = x^4$ from first principles.

$f(x+h) = (x+h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$

$$\frac{f(x+h) - f(x)}{h} = \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h} = 4x^3 + 6x^2h + 4xh^2 + h^3$$

As $h \to 0$, all terms involving h tend to zero.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$f'(x) = 4x^3$

End of Worked Solutions