Question 1 (Jun 2009, Q12)

Worked Solution

(i) Gradient of chord $= \dfrac{(3.1^2 - 7) - (3^2 - 7)}{3.1 - 3} = \dfrac{9.61 - 7 - 9 + 7}{0.1} = \dfrac{0.61}{0.1}$

Gradient $= 6.1$

(ii) With $f(x) = x^2 - 7$:

$$\frac{f(3+h) - f(3)}{h} = \frac{(3+h)^2 - 7 - (3^2 - 7)}{h} = \frac{9 + 6h + h^2 - 7 - 9 + 7}{h} = \frac{6h + h^2}{h}$$

$= 6 + h$

(iii) As $h \to 0$, the expression $6 + h \to 6$. The gradient of the chord approaches the gradient of the tangent, so:

Gradient at $x = 3$ is 6.

(iv) At $x = 3$: $y = 9 - 7 = 2$, gradient $= 6$.

Tangent: $y - 2 = 6(x - 3)$

$y = 6x - 16$

(v) Tangent $y = 6x - 16$ meets x-axis when $y = 0$: $x = \frac{16}{6} = \frac{8}{3}$, so $P = \left(\frac{8}{3}, 0\right)$.

Curve $y = x^2 - 7$ meets positive x-axis when $x^2 = 7$, so $Q = (\sqrt{7}, 0)$.

$$PQ = \left|\sqrt{7} - \frac{8}{3}\right| = \sqrt{7} - \frac{8}{3} \approx 2.6458 - 2.6667$$

$PQ \approx 0.021$

Question 2 (Jan 2007, Q5)

Worked Solution

(i) $y = \dfrac{4}{x^2}$. At $A(2,1)$: $y_A = \frac{4}{4} = 1$. ✓

At B, $x = 2.1$: $y_B = \dfrac{4}{2.1^2} = \dfrac{4}{4.41} \approx 0.9070$.

Gradient of $AB = \dfrac{0.9070 - 1}{2.1 - 2} = \dfrac{-0.0930}{0.1}$

≈ -0.93

(ii) A point C closer to A than B gives a better approximation, e.g. any x-coordinate strictly between 1.91 and 2 (or between 2 and 2.1, exclusive of 2.1).

Any x strictly between 1.91 and 2, or strictly between 2 and 2.1 (not 2.1 itself).

(iii) Write $y = 4x^{-2}$, so $\dfrac{dy}{dx} = -8x^{-3}$.

At $x = 2$: gradient $= -\dfrac{8}{8}$

Gradient at $A = -1$

Question 3 (Jun 2010, Q10)

Worked Solution

(i) $y = x^4$, so $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 4x^3$.

At $x = 2$: gradient $= 4 \times 8 = 32$. When $x = 2$, $y = 16$.

Tangent: $y - 16 = 32(x - 2)$

$y = 32x - 48$

(ii) Gradient of chord $= \dfrac{2.1^4 - 2^4}{2.1 - 2} = \dfrac{19.4481 - 16}{0.1} = \dfrac{3.4481}{0.1}$

$= 34.481$

(iii)(A) $(2 + h)^4 = 2^4 + 4 \cdot 2^3 h + 6 \cdot 2^2 h^2 + 4 \cdot 2h^3 + h^4$

$= 16 + 32h + 24h^2 + 8h^3 + h^4$

(iii)(B)

$$\frac{(2 + h)^4 - 2^4}{h} = \frac{32h + 24h^2 + 8h^3 + h^4}{h}$$

$= 32 + 24h + 8h^2 + h^3$

(iii)(C) As $h \to 0$, the expression $32 + 24h + 8h^2 + h^3 \to 32$.

The gradient of the chord approaches the gradient of the tangent at $x = 2$, which is the limit of the gradient of the chord. Therefore the gradient of $y = x^4$ at $x = 2$ is 32.

Question 4 (OCR H230/02, Sample Q7)

Worked Solution

Differentiate $f(x) = x^4$ from first principles.

$f(x+h) = (x+h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$

$$\frac{f(x+h) - f(x)}{h} = \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h} = 4x^3 + 6x^2h + 4xh^2 + h^3$$

As $h \to 0$, all terms involving h tend to zero.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$f'(x) = 4x^3$

Question 5 (Jun 2016, Q10)

Worked Solution

(i) $y = x^2 - 2x$. At $x = 5$: $y = 25 - 10 = 15$. At $x = 5.1$: $y = 26.01 - 10.2 = 15.81$.

Gradient of chord $= \dfrac{15.81 - 15}{5.1 - 5} = \dfrac{0.81}{0.1}$

$= 8.1$

(ii) With $f(x) = x^2 - 2x$:

$$\frac{f(5+h) - f(5)}{h} = \frac{(5+h)^2 - 2(5+h) - 15}{h} = \frac{25 + 10h + h^2 - 10 - 2h - 15}{h} = \frac{8h + h^2}{h}$$

$= 8 + h$

(iii) As $h \to 0$, $8 + h \to 8$. The gradient of the chord approaches the gradient of the curve, so:

Gradient at $x = 5$ is 8.

(iv) At $x = 5$: $y = 15$, gradient $= 8$. Tangent: $y - 15 = 8(x - 5)$, i.e. $y = 8x - 25$.

x-intercept: $0 = 8x - 25 \Rightarrow x = \frac{25}{8} = 3.125$. y-intercept: $x = 0 \Rightarrow y = -25$.

Area of triangle $= \frac{1}{2} \times \frac{25}{8} \times 25 = \frac{625}{16}$

Area $= \dfrac{625}{16} = 39.0625$

Question 6 (OCR 4721, Jun 2016, Q8)

Worked Solution

(i) $y = 2x^2$. At A, $x = 5$: $y_1 = 2(25) = 50$. At B, $x = 5 + h$: $y_2 = 2(5 + h)^2 = 2(25 + 10h + h^2) = 50 + 20h + 2h^2$.

Gradient of $AB = \dfrac{(50 + 20h + 2h^2) - 50}{(5 + h) - 5} = \dfrac{20h + 2h^2}{h}$

$= 20 + 2h$

(ii) As $h \to 0$, the gradient of AB approaches 20. Since the gradient of the chord AB approaches the gradient of the curve at A as $h \to 0$, the gradient of the curve at A is 20.

(iii) Gradient of curve at A is 20, so gradient of normal at $A = -\dfrac{1}{20}$.

Normal passes through $A(5, 50)$:

$$y - 50 = -\tfrac{1}{20}(x - 5)$$

At $x = 0$: $y = 50 - \frac{1}{20}(-5) = 50 + \frac{1}{4}$

y-coordinate of $C = 50\frac{1}{4}$

Question 7 (Edexcel 8MA0/01, Sample Assessment 1, Q6)

Worked Solution

Prove from first principles that the derivative of $3x^2$ is $6x$.

Consider $\dfrac{3(x+h)^2 - 3x^2}{h}$.

Expand: $3(x+h)^2 = 3x^2 + 6xh + 3h^2$.

$$\frac{3x^2 + 6xh + 3h^2 - 3x^2}{h} = \frac{6xh + 3h^2}{h} = 6x + 3h$$

As $h \to 0$, $6x + 3h \to 6x$.

Therefore the derivative of $3x^2$ is $6x$.