Question 1

Worked Solution

Find $\int (8x^3 + 4)\,\mathrm{d}x$.

$$\int (8x^3 + 4)\,\mathrm{d}x = \frac{8x^4}{4} + 4x + c = 2x^4 + 4x + c$$

$2x^4 + 4x + c$

Question 2

Worked Solution

Find $\int \left(2x^4 - \frac{4}{\sqrt{x}} + 3\right) \mathrm{d}x$.

Write $\frac{4}{\sqrt{x}} = 4x^{-\frac{1}{2}}$:

$$\int \left(2x^4 - 4x^{-\frac{1}{2}} + 3\right) \mathrm{d}x = \frac{2x^5}{5} - \frac{4x^{\frac{1}{2}}}{\frac{1}{2}} + 3x + c = \frac{2}{5}x^5 - 8x^{\frac{1}{2}} + 3x + c$$

$\frac{2}{5}x^5 - 8\sqrt{x} + 3x + c$

Question 3

Worked Solution

Find $\int \left(2x^5 - \frac{1}{4x^3} - 5\right) \mathrm{d}x$.

Write $\frac{1}{4x^3} = \frac{1}{4}x^{-3}$:

$$\int \left(2x^5 - \tfrac{1}{4}x^{-3} - 5\right) \mathrm{d}x = \frac{2x^6}{6} - \frac{\frac{1}{4}x^{-2}}{-2} - 5x + c = \frac{x^6}{3} + \frac{1}{8}x^{-2} - 5x + c$$

$$\frac{1}{3}x^6 + \frac{1}{8x^2} - 5x + c$$

Question 4

Worked Solution

Find $\int \left(3x^2 - \frac{4}{x^2}\right)\,\mathrm{d}x$.

Write $\frac{4}{x^2} = 4x^{-2}$:

$$\int \left(3x^2 - 4x^{-2}\right)\mathrm{d}x = x^3 - \frac{4x^{-1}}{-1} + c = x^3 + \frac{4}{x} + c$$

$x^3 + \frac{4}{x} + c$

Question 5

Worked Solution

$y = 2x^5 + \frac{6}{\sqrt{x}} = 2x^5 + 6x^{-\frac{1}{2}}$, $x > 0$.

(a) Differentiate:

$$\frac{dy}{dx} = 10x^4 + 6 \cdot \left(-\tfrac{1}{2}\right) x^{-\frac{3}{2}} = 10x^4 - 3x^{-\frac{3}{2}}$$

$$\frac{dy}{dx} = 10x^4 - 3x^{-\frac{3}{2}}$$

(b) Integrate y:

$$\int y \, dx = \frac{2x^6}{6} + \frac{6x^{\frac{1}{2}}}{\frac{1}{2}} + c = \frac{x^6}{3} + 12x^{\frac{1}{2}} + c$$

$$\frac{x^6}{3} + 12\sqrt{x} + c$$

Question 6

Worked Solution

$y = 2x^5 + 7 + \frac{1}{x^3} = 2x^5 + 7 + x^{-3}$, $x \neq 0$.

(a) Differentiate:

$$\frac{dy}{dx} = 10x^4 - 3x^{-4}$$

$$\frac{dy}{dx} = 10x^4 - \frac{3}{x^4}$$

(b) Integrate y:

$$\int y \, dx = \frac{2x^6}{6} + 7x + \frac{x^{-2}}{-2} + c = \frac{x^6}{3} + 7x - \frac{1}{2x^2} + c$$

$$\frac{x^6}{3} + 7x - \frac{1}{2x^2} + c$$

Question 7

Worked Solution

$y = 3x^2 + 4\sqrt{x} = 3x^2 + 4x^{\frac{1}{2}}$, $x > 0$.

(a) Differentiate:

$$\frac{dy}{dx} = 6x + 4 \cdot \tfrac{1}{2}x^{-\frac{1}{2}} = 6x + 2x^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = 6x + \frac{2}{\sqrt{x}}$$

(b) Differentiate again:

$$\frac{d^2y}{dx^2} = 6 + 2 \cdot \left(-\tfrac{1}{2}\right) x^{-\frac{3}{2}} = 6 - x^{-\frac{3}{2}}$$

$$\frac{d^2y}{dx^2} = 6 - x^{-\frac{3}{2}}$$

(c) Integrate y:

$$\int y \, dx = x^3 + \frac{4x^{\frac{3}{2}}}{\frac{3}{2}} + c = x^3 + \frac{8}{3}x^{\frac{3}{2}} + c$$

$$x^3 + \frac{8}{3}x^{\frac{3}{2}} + c$$

Question 8

Worked Solution

Evaluate $\displaystyle\int_1^8 \frac{1}{\sqrt{x}}\,\mathrm{d}x$, giving your answer in the form $a + b\sqrt{2}$.

Write $\dfrac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$:

$$\int_1^8 x^{-\frac{1}{2}}\,\mathrm{d}x = \left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^8 = \left[2x^{\frac{1}{2}}\right]_1^8 = 2\sqrt{8} - 2\sqrt{1} = 4\sqrt{2} - 2$$

$-2 + 4\sqrt{2}$ (i.e. $a = -2$, $b = 4$)

Question 9

Worked Solution

Evaluate $\displaystyle\int_1^{\sqrt{3}} \left(\frac{x^3}{6} + \frac{1}{3x^2}\right) \mathrm{d}x$, giving the answer in the form $a + b\sqrt{3}$.

Write $\dfrac{1}{3x^2} = \dfrac{1}{3}x^{-2}$:

$$\int \left(\frac{x^3}{6} + \frac{1}{3}x^{-2}\right) \mathrm{d}x = \frac{x^4}{24} + \frac{x^{-1}}{3(-1)} = \frac{x^4}{24} - \frac{1}{3x}$$

Apply limits:

$$\left[\frac{x^4}{24} - \frac{1}{3x}\right]_1^{\sqrt{3}} = \left(\frac{9}{24} - \frac{1}{3\sqrt{3}}\right) - \left(\frac{1}{24} - \frac{1}{3}\right)$$

$$= \frac{9}{24} - \frac{1}{3\sqrt{3}} - \frac{1}{24} + \frac{1}{3} = \frac{8}{24} + \frac{1}{3} - \frac{1}{3\sqrt{3}}$$

$$= \frac{1}{3} + \frac{1}{3} - \frac{\sqrt{3}}{9} = \frac{2}{3} - \frac{\sqrt{3}}{9}$$

$a = \dfrac{2}{3}$, $b = -\dfrac{1}{9}$, i.e. $\dfrac{2}{3} - \dfrac{1}{9}\sqrt{3}$

Question 10

Worked Solution

k is a positive constant and $\int_1^k \left(\frac{5}{2\sqrt{x}} + 3\right) \mathrm{d}x = 4$.

(a) Integrate $\frac{5}{2\sqrt{x}} + 3 = \frac{5}{2}x^{-\frac{1}{2}} + 3$:

$$\left[5x^{\frac{1}{2}} + 3x\right]_1^k = (5\sqrt{k} + 3k) - (5 + 3) = 5\sqrt{k} + 3k - 8$$

Set equal to 4:

$$5\sqrt{k} + 3k - 8 = 4 \implies 3k + 5\sqrt{k} - 12 = 0$$

$3k + 5\sqrt{k} - 12 = 0$ (as required)

(b) Let $u = \sqrt{k}$:

$$3u^2 + 5u - 12 = 0 \implies (3u - 4)(u + 3) = 0$$

$$u = \frac{4}{3} \quad \text{or} \quad u = -3$$

Since $k > 0$ we need $\sqrt{k} = \frac{4}{3}$, so:

$$k = \frac{16}{9}$$

$k = \frac{16}{9}$

Question 11

Worked Solution

$\mathrm{f}(x) = 2x + 3 + \dfrac{12}{x^2} = 2x + 3 + 12x^{-2}$, $x > 0$.

Show that $\displaystyle\int_1^{2\sqrt{2}} \mathrm{f}(x)\,\mathrm{d}x = 16 + 3\sqrt{2}$.

Integrate:

$$\int \mathrm{f}(x)\,\mathrm{d}x = x^2 + 3x - \frac{12}{x}$$

Apply limits (1 to $2\sqrt{2}$):

Upper ($x = 2\sqrt{2}$):

$$(2\sqrt{2})^2 + 3(2\sqrt{2}) - \frac{12}{2\sqrt{2}} = 8 + 6\sqrt{2} - \frac{6}{\sqrt{2}} = 8 + 6\sqrt{2} - \frac{6\sqrt{2}}{2} = 8 + 6\sqrt{2} - 3\sqrt{2} = 8 + 3\sqrt{2}$$

Lower ($x = 1$):

$$1 + 3 - 12 = -8$$

Result:

$$(8 + 3\sqrt{2}) - (-8) = 16 + 3\sqrt{2} \qquad \checkmark$$

$$\int_1^{2\sqrt{2}} \mathrm{f}(x)\,\mathrm{d}x = 16 + 3\sqrt{2} \quad \text{(shown)}$$

Question 12

Worked Solution

(a) Find $\int \left(\frac{4}{x^3} + kx\right) \mathrm{d}x$ where k is a constant.

Write $\frac{4}{x^3} = 4x^{-3}$:

$$\int \left(4x^{-3} + kx\right) \mathrm{d}x = \frac{4x^{-2}}{-2} + \frac{kx^2}{2} + c = -\frac{2}{x^2} + \frac{k}{2}x^2 + c$$

$-\frac{2}{x^2} + \frac{k}{2}x^2 + c$

(b) Given $\int_{0.5}^{2} \left(\frac{4}{x^3} + kx\right) \mathrm{d}x = 8$:

$$\left[-\frac{2}{x^2} + \frac{k}{2}x^2\right]_{0.5}^{2} = \left(-\frac{2}{4} + \frac{4k}{2}\right) - \left(-\frac{2}{0.25} + \frac{k}{2}(0.25)\right)$$
$$= \left(-\frac{1}{2} + 2k\right) - \left(-8 + \frac{k}{8}\right) = -\frac{1}{2} + 2k + 8 - \frac{k}{8} = \frac{15}{2} + \frac{15k}{8}$$

Set equal to 8:

$$\frac{15}{2} + \frac{15k}{8} = 8 \implies \frac{15k}{8} = \frac{1}{2} \implies k = \frac{4}{15}$$

$k = \frac{4}{15}$

Question 13

Worked Solution

Calculate $\lim_{\delta x \to 0} \sum_{x=4}^{9} \sqrt{x}\,\delta x$.

This limit equals the definite integral:

$$\int_4^9 \sqrt{x}\,\mathrm{d}x = \int_4^9 x^{\frac{1}{2}}\,\mathrm{d}x = \left[\frac{2}{3}x^{\frac{3}{2}}\right]_4^9 = \frac{2}{3}(27) - \frac{2}{3}(8) = 18 - \frac{16}{3} = \frac{54-16}{3} = \frac{38}{3}$$

$$\frac{38}{3}$$

End of Worked Solutions