

Question 1 (Jun 2010, Q6a)**Worked Solution**

Find the exact area enclosed by $y = x^2 + 4x$, the x -axis, $x = 3$ and $x = 5$.

$$\begin{aligned}\int_3^5 (x^2 + 4x) \, dx &= \left[\frac{x^3}{3} + 2x^2 \right]_3^5 \\ &= \left(\frac{125}{3} + 50 \right) - (9 + 18) \\ &= \frac{125}{3} + 50 - 27 \\ &= \frac{125}{3} + 23 \\ &= \frac{194}{3}\end{aligned}$$

$$\text{Area} = \frac{194}{3} = 64\frac{2}{3}$$

Question 2 (Jun 2006, Q4)

Worked Solution

Curve: $y = 4 - x^2$. Line: $y = x + 2$.

(i) Find x -coordinates of intersection:

$$4 - x^2 = x + 2 \implies x^2 + x - 2 = 0 \implies (x + 2)(x - 1) = 0$$

$$x = -2 \text{ and } x = 1$$

(ii) Area of shaded region (between line and curve):

$$\int_{-2}^1 [(4 - x^2) - (x + 2)] dx = \int_{-2}^1 (2 - x - x^2) dx = \left[2x - \frac{x^2}{2} - \frac{x^3}{3} \right]_{-2}^1$$

$$\text{Upper } (x = 1): 2 - \frac{1}{2} - \frac{1}{3} = \frac{12-3-2}{6} = \frac{7}{6}$$

$$\text{Lower } (x = -2): -4 - 2 + \frac{8}{3} = -6 + \frac{8}{3} = \frac{-18+8}{3} = -\frac{10}{3}$$

$$\text{Area} = \frac{7}{6} - \left(-\frac{10}{3}\right) = \frac{7}{6} + \frac{20}{6} = \frac{27}{6} = \frac{9}{2} = 4\frac{1}{2}$$

$$\text{Area} = 4\frac{1}{2}$$

Question 3 (Jan 2006, Q8)

Worked Solution

$f(x) = 2x^3 + kx^2 - x + 6$; $(x + 1)$ is a factor.

(i) Show $k = -5$ and factorise $f(x)$ completely.

$$f(-1) = -2 + k + 1 + 6 = k + 5 = 0 \Rightarrow k = -5.$$

$f(x) = 2x^3 - 5x^2 - x + 6$. Divide by $(x + 1)$: $f(x) = (x + 1)(2x^2 - 7x + 6) = (x + 1)(2x - 3)(x - 2)$.

$$k = -5; f(x) = (x + 1)(2x - 3)(x - 2)$$

(ii) Find $\int_{-1}^2 f(x) dx$:

$$\begin{aligned} \int (2x^3 - 5x^2 - x + 6) dx &= \frac{x^4}{2} - \frac{5x^3}{3} - \frac{x^2}{2} + 6x \\ &= \left(\frac{16}{2} - \frac{40}{3} - 2 + 12 \right) - \left(\frac{1}{2} + \frac{5}{3} - \frac{1}{2} - 6 \right) = \left(18 - \frac{40}{3} \right) - \left(\frac{5}{3} - 6 \right) \\ &= 18 - \frac{40}{3} - \frac{5}{3} + 6 = 24 - 15 = 9 \end{aligned}$$

$$\int_{-1}^2 f(x) dx = 9$$

(iii) The value of 9 does not equal the area because the curve crosses the x -axis within $[-1, 2]$ (at $x = \frac{3}{2}$), so part of the region lies below the x -axis and contributes negatively to the integral. The actual area requires evaluating the integrals over $[-1, \frac{3}{2}]$ and $[\frac{3}{2}, 2]$ separately and summing the absolute values.

Question 4 (Jan 2012, Q7b)**Worked Solution**

Curves: $y = 6x^{\frac{3}{2}}$ and $y = \frac{8}{x^2} - 2$, intersecting at $(1, 6)$. Find area enclosed by both curves and the x -axis.

Region from $x = 0$ to $x = 1$ (under $y = 6x^{\frac{3}{2}}$):

$$\int_0^1 6x^{\frac{3}{2}} dx = \left[\frac{6x^{\frac{5}{2}}}{\frac{5}{2}} \right]_0^1 = \frac{12}{5}$$

Region from $x = 1$ to $x = 2$ (under $y = \frac{8}{x^2} - 2 = 8x^{-2} - 2$; note this curve meets x -axis at $x = 2$):

$$\int_1^2 \left(\frac{8}{x^2} - 2 \right) dx = \left[-\frac{8}{x} - 2x \right]_1^2 = (-4 - 4) - (-8 - 2) = -8 - (-10) = 2$$

$$\text{Total area} = \frac{12}{5} + 2 = \frac{22}{5}$$

$$\text{Area} = \frac{22}{5}$$

Question 5 (Jan 2008, Q7)

Worked Solution

Curve: $y = x^2 - 3x$. Line: $x = 5$.

(i) $\int_0^5 (x^2 - 3x) dx$ does not give the total shaded area because the curve is below the x -axis for $0 < x < 3$, so that part contributes negatively to the integral.

(ii) Find exact total area:

Region below x -axis $[0, 3]$:

$$\left| \int_0^3 (x^2 - 3x) dx \right| = \left| \left[\frac{x^3}{3} - \frac{3x^2}{2} \right]_0^3 \right| = \left| 9 - \frac{27}{2} \right| = \left| -\frac{9}{2} \right| = \frac{9}{2}$$

Region above x -axis $[3, 5]$:

$$\begin{aligned} \int_3^5 (x^2 - 3x) dx &= \left[\frac{x^3}{3} - \frac{3x^2}{2} \right]_3^5 = \left(\frac{125}{3} - \frac{75}{2} \right) - \left(9 - \frac{27}{2} \right) = \frac{125}{3} - \frac{75}{2} - 9 + \frac{27}{2} \\ &= \frac{125}{3} - 24 - 9 = \frac{125}{3} - \frac{24}{1} = \frac{125 - 72}{3} \dots \end{aligned}$$

$$\begin{aligned} \text{More carefully: } &= \frac{125}{3} - \frac{75}{2} + \frac{27}{2} - 9 = \frac{125}{3} - 24 - 9 = \frac{125}{3} - \frac{48}{2} \\ &= \frac{125}{3} - \frac{75-27}{2} - 9 = \frac{125}{3} - 24 - 9 \end{aligned}$$

$$\text{Let me compute step by step: } \frac{125}{3} - \frac{75}{2} - 9 + \frac{27}{2} = \frac{125}{3} + \frac{27-75}{2} - 9 = \frac{125}{3} - 24 - 9 = \frac{125}{3} - \frac{99}{3} \dots$$

$$-9 = -\frac{27}{3}. \text{ So: } \frac{125-27}{3} - \frac{75-27}{2} = \frac{98}{3} - 24 = \frac{98-72}{3} = \frac{26}{3}$$

$$\text{Check: } \frac{125}{3} - \frac{75}{2} - 9 + \frac{27}{2}. \text{ LCD} = 6: \frac{250}{6} - \frac{225}{6} - \frac{54}{6} + \frac{81}{6} = \frac{250-225-54+81}{6} = \frac{52}{6} = \frac{26}{3}. \checkmark$$

$$\text{Total area} = \frac{9}{2} + \frac{26}{3} = \frac{27}{6} + \frac{52}{6} = \frac{79}{6} = 13\frac{1}{6}$$

$$\text{Total area} = 13\frac{1}{6}$$

Question 6 (Jan 2007, Q10)

Worked Solution

$$y = 1 - 3x^{-\frac{1}{2}}.$$

(i) Verify intersection with x -axis at $(9, 0)$: $y = 1 - \frac{3}{\sqrt{9}} = 1 - 1 = 0 \checkmark$

(ii) Area enclosed by curve, x -axis and $x = a$ ($a > 9$) is 4 square units. Find a .

$$\int_9^a \left(1 - 3x^{-\frac{1}{2}}\right) dx = \left[x - 6x^{\frac{1}{2}}\right]_9^a = (a - 6\sqrt{a}) - (9 - 18) = a - 6\sqrt{a} + 9$$

Set equal to 4:

$$a - 6\sqrt{a} + 9 = 4 \implies a - 6\sqrt{a} + 5 = 0$$

Let $u = \sqrt{a}$: $u^2 - 6u + 5 = 0 \implies (u - 1)(u - 5) = 0 \implies u = 1$ or $u = 5$.

Since $a > 9$: $\sqrt{a} = 5$, so $a = 25$.

$$a = 25$$

Question 7 (Jan 2011, Q9)

Worked Solution

$$f(x) = -4x^3 + 9x^2 + 10x - 3.$$

(i) Verify $f(3) = 0$: $f(3) = -108 + 81 + 30 - 3 = 0 \checkmark$. Factor: $(x - 3)$.

(ii) Express as product of linear and quadratic factor: $-4x^3 + 9x^2 + 10x - 3 = (x - 3)(-4x^2 - 3x + 1)$

$$(x - 3)(-4x^2 - 3x + 1)$$

(iii) Other intersections: $-4x^2 - 3x + 1 = 0 \Rightarrow 4x^2 + 3x - 1 = 0 \Rightarrow (1 - 4x)(x + 1) = 0$

$$x = \frac{1}{4} \text{ and } x = -1$$

(iv) Total area enclosed by curve and x -axis. Roots at $x = -1$, $x = \frac{1}{4}$, $x = 3$.

Curve is positive on $(-1, \frac{1}{4})$ and negative on $(\frac{1}{4}, 3)$.

$$\int f(x) dx = -x^4 + 3x^3 + 5x^2 - 3x$$

$$\text{Region } [-1, \frac{1}{4}]: F(\frac{1}{4}) = -\frac{1}{256} + \frac{3}{64} + \frac{5}{16} - \frac{3}{4} = \frac{-1+12+80-192}{256} = \frac{-101}{256}$$

$$F(-1) = -1 - 3 + 5 + 3 = 4$$

$$\int_{-1}^{\frac{1}{4}} = -\frac{101}{256} - 4 = -\frac{1125}{256} \dots$$

The curve at $(-1, \frac{1}{4})$ is actually positive (above x -axis). Let me verify: $f(0) = -3 < 0$. So the curve is negative between the roots -1 and $\frac{1}{4}$!

Actually: $f(x) = (x - 3)(-4x^2 - 3x + 1)$. At $x = 0$: $(0 - 3)(-0 - 0 + 1) = (-3)(1) = -3 < 0$. So curve is negative on $(-1, \frac{1}{4})$ and positive on $(\frac{1}{4}, 3)$.

Region $[-1, \frac{1}{4}]$ (below axis): $|F(\frac{1}{4}) - F(-1)|$

$$F(\frac{1}{4}) = -\frac{1}{256} + 3 \cdot \frac{1}{64} + 5 \cdot \frac{1}{16} - 3 \cdot \frac{1}{4} = -\frac{1}{256} + \frac{12}{256} + \frac{80}{256} - \frac{192}{256} = \frac{-101}{256}$$

$$F(-1) = -(-1)^4 + 3(-1)^3 + 5(-1)^2 - 3(-1) = -1 - 3 + 5 + 3 = 4$$

$$\int_{-1}^{\frac{1}{4}} = -\frac{101}{256} - 4 = -\frac{1125}{256} \text{ (negative, consistent). Area} = \frac{1125}{256}.$$

Region $[\frac{1}{4}, 3]$ (above axis): $F(3) = -81 + 81 + 45 - 9 = 36$

$$\int_{\frac{1}{4}}^3 = 36 - (-\frac{101}{256}) = 36 + \frac{101}{256} = \frac{9216+101}{256} = \frac{9317}{256}$$

$$\text{Total area} = \frac{1125}{256} + \frac{9317}{256} = \frac{10442}{256} = \frac{5221}{128} \approx 40.8$$

$$\text{Total area} = \frac{5221}{128} \approx 40.8$$

Question 8 (Jan 2016, Q7)

Worked Solution

$$f(x) = x^3 - 3x^2 - x + 3.$$

(i) Find quotient and remainder when $f(x)$ is divided by $(x + 1)$:

Long division of $x^3 - 3x^2 - x + 3$ by $(x + 1)$:

$$x^3 - 3x^2 - x + 3 = (x + 1)(x^2 - 4x + 3) + 0$$

Quotient: $x^2 - 4x + 3$, Remainder: 0.

$$\text{Quotient} = x^2 - 4x + 3, \text{ Remainder} = 0$$

(ii) Three roots of $f(x) = 0$:

$$(x + 1)(x^2 - 4x + 3) = (x + 1)(x - 1)(x - 3) = 0$$

$$x = -1, 1, 3$$

Curve C : $y = x^4 - 4x^3 - 2x^2 + 12x + 9$.

(iii) Show stationary points satisfy $x^3 - 3x^2 - x + 3 = 0$:

$$\frac{dy}{dx} = 4x^3 - 12x^2 - 4x + 12 = 4(x^3 - 3x^2 - x + 3) = 0 \implies x^3 - 3x^2 - x + 3 = 0 \quad \checkmark$$

(iv) Area enclosed by C and x -axis (one arch from $x = -1$ to $x = 1$, above axis). By symmetry or direct computation:

$$\int_{-1}^1 (x^4 - 4x^3 - 2x^2 + 12x + 9) dx = \left[\frac{x^5}{5} - x^4 - \frac{2x^3}{3} + 6x^2 + 9x \right]_{-1}^1$$

$$\text{Upper } (x = 1): \frac{1}{5} - 1 - \frac{2}{3} + 6 + 9 = \frac{1}{5} - \frac{2}{3} + 14 = \frac{3-10}{15} + 14 = -\frac{7}{15} + 14 = \frac{203}{15}$$

$$\text{Lower } (x = -1): -\frac{1}{5} - 1 + \frac{2}{3} + 6 - 9 = -\frac{1}{5} + \frac{2}{3} - 4 = \frac{-3+10}{15} - 4 = \frac{7}{15} - 4 = -\frac{53}{15}$$

$$\text{Area} = \frac{203}{15} - \left(-\frac{53}{15}\right) = \frac{256}{15}$$

$$\text{Area} = \frac{256}{15}$$

Question 9 (Jan 2015, Q6)

Worked Solution

$$f(x) = x^3 - 19x + 30.$$

(i) Express as product of 3 linear factors ($x = 2$ is a root): $(x - 2)(x^2 + 2x - 15) = (x - 2)(x + 5)(x - 3)$

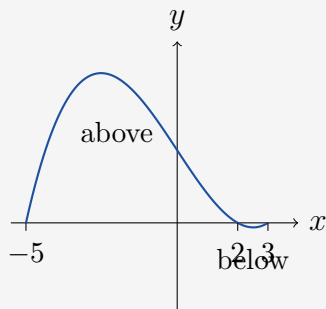
$$f(x) = (x - 2)(x + 5)(x - 3)$$

(ii) Find $\int_{-5}^3 f(x) dx$:

$$\begin{aligned} \int (x^3 - 19x + 30) dx &= \frac{x^4}{4} - \frac{19x^2}{2} + 30x \\ &= \left(\frac{81}{4} - \frac{171}{2} + 90 \right) - \left(\frac{625}{4} - \frac{475}{2} - 150 \right) \\ &= \frac{81 - 342 + 360}{4} - \frac{625 - 950 - 600}{4} = \frac{99}{4} - \frac{-925}{4} = \frac{1024}{4} = 256 \end{aligned}$$

$$\int_{-5}^3 f(x) dx = 256$$

(iii) This does not equal the area enclosed because the curve crosses the x -axis at $x = 2$ within $[-5, 3]$. The region from $x = 2$ to $x = 3$ lies below the x -axis and contributes negatively to the integral. The actual area requires treating the sub-intervals separately.



End of Worked Solutions