Question 1

Worked Solution

Curve C: $y = (1+x)(4-x)$; intersects x-axis at $x=-1$ and $x=4$. Find area of region R.

Expand: $(1+x)(4-x) = 4 - x + 4x - x^2 = 4 + 3x - x^2$.

$$\int_{-1}^{4} (4 + 3x - x^2)\,\mathrm{d}x = \left[4x + \frac{3x^2}{2} - \frac{x^3}{3}\right]_{-1}^{4}$$

Upper ($x=4$): $16 + 24 - \dfrac{64}{3} = 40 - \dfrac{64}{3} = \dfrac{56}{3}$

Lower ($x=-1$): $-4 + \dfrac{3}{2} + \dfrac{1}{3} = -\dfrac{24}{6} + \dfrac{9}{6} + \dfrac{2}{6} = -\dfrac{13}{6}$

Area $= \dfrac{56}{3} - \left(-\dfrac{13}{6}\right) = \dfrac{112}{6} + \dfrac{13}{6} = \dfrac{125}{6}$

Area $= \dfrac{125}{6}$

Question 2

Worked Solution

Curve: $y = 3x - x^{\frac{3}{2}}$, $x \geq 0$. Region S bounded by curve and x-axis.

(a) Find $\int \left(3x - x^{\frac{3}{2}}\right) \mathrm{d}x$.

$$= \frac{3x^2}{2} - \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + c = \frac{3x^2}{2} - \frac{2}{5}x^{\frac{5}{2}} + c$$

$$\frac{3x^2}{2} - \frac{2}{5}x^{\frac{5}{2}} + c$$

(b) Find where curve meets x-axis ($y = 0$, $x \geq 0$):

$$3x - x^{\frac{3}{2}} = 0 \implies x(3 - x^{\frac{1}{2}}) = 0 \implies x = 0 \text{ or } x = 9$$

Area of S:

$$\int_0^9 \left(3x - x^{\frac{3}{2}}\right) \mathrm{d}x = \left[\frac{3x^2}{2} - \frac{2}{5}x^{\frac{5}{2}}\right]_0^9 = \frac{3(81)}{2} - \frac{2}{5}(243) - 0 = \frac{243}{2} - \frac{486}{5} = \frac{1215 - 972}{10} = \frac{243}{10}$$

$$\text{Area} = \frac{243}{10}$$

Question 3

Worked Solution

Line $y = 10$ cuts curve $y = x^2 + 2x + 2$ at A and B.

(a) Find x-coordinates of A and B:

$$x^2 + 2x + 2 = 10 \implies x^2 + 2x - 8 = 0 \implies (x+4)(x-2) = 0$$

$x = -4$ and $x = 2$

(b) Area of R (between line and curve):

Method: Area under line − Area under curve from -4 to 2.

Area under line $= 10 \times (2 - (-4)) = 60$.

Area under curve:

$$\int_{-4}^{2} (x^2 + 2x + 2)\,\mathrm{d}x = \left[\frac{x^3}{3} + x^2 + 2x\right]_{-4}^{2}$$

$$= \left(\frac{8}{3} + 4 + 4\right) - \left(\frac{-64}{3} + 16 - 8\right) = \frac{8}{3} + 8 - \left(\frac{-64}{3} + 8\right) = \frac{8}{3} + 8 + \frac{64}{3} - 8 = \frac{72}{3} = 24$$

Area of $R = 60 - 24 = 36$.

Area $= 36$

Question 4

Worked Solution

$y = 2x^3 - 17x^2 + 40x$. Show area of region R (from $x = 0$ to minimum at $x = k$) is $\dfrac{256}{3}$.

Find k (minimum turning point):

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 - 34x + 40 = 0 \implies 2(3x-5)(x-4) = 0 \implies x = \tfrac{5}{3} \text{ or } x = 4$$

Second derivative test: $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12x - 34$; at $x = 4$: $48 - 34 = 14 > 0$ (minimum). So $k = 4$.

Area:

$$\int_0^4 (2x^3 - 17x^2 + 40x)\,\mathrm{d}x = \left[\frac{x^4}{2} - \frac{17x^3}{3} + 20x^2\right]_0^4$$

$$= \frac{256}{2} - \frac{17(64)}{3} + 20(16) = 128 - \frac{1088}{3} + 320 = 448 - \frac{1088}{3} = \frac{1344-1088}{3} = \frac{256}{3} \quad \checkmark$$

$$\text{Area} = \frac{256}{3} \quad \text{(shown)}$$

Question 5

Worked Solution

Curve: $y = x(x+2)(x-4) = x^3 - 2x^2 - 8x$.

(a) Show area of R_1 (between curve and negative x-axis, $x \in [-2, 0]$) is $\frac{20}{3}$.

The curve is negative on $[-2, 0]$, so area $= -\int_{-2}^{0}$.

$$\int_{-2}^{0}(x^3-2x^2-8x)\,\mathrm{d}x = \left[\frac{x^4}{4} - \frac{2x^3}{3} - 4x^2\right]_{-2}^{0} = 0-\left(4+\frac{16}{3}-16\right) = -\left(-12+\frac{16}{3}\right) = 12-\frac{16}{3} = \frac{20}{3}$$

Area of $R_1 = \frac{20}{3}$ (shown)

(b) Area of R_2 (from $x = 0$ to $x = b$, $0 < b < 4$) equals $\frac{20}{3}$:

$$\int_0^b (x^3 - 2x^2 - 8x)\,\mathrm{d}x = -\frac{20}{3}$$

(negative since curve is below x-axis for $0 < x < 4$)

$$\frac{b^4}{4} - \frac{2b^3}{3} - 4b^2 = -\frac{20}{3}$$

Multiply by 12: $3b^4 - 8b^3 - 48b^2 + 80 = 0$

Factor as $(b+2)^2(3b^2-20b+20) = 0$: expand the right side: $(b^2+4b+4)(3b^2-20b+20) = 3b^4 - 20b^3 + 20b^2 + 12b^3 - 80b^2 + 80b + 12b^2 - 80b + 80 = 3b^4 - 8b^3 - 48b^2 + 0b + 80$ ✓

$(b+2)^2(3b^2 - 20b + 20) = 0$ (verified)

(c) The root $b = 5.442$ lies outside the interval $(0, 4)$; it is in the region $x > 4$ where the curve is above the x-axis. At $b = 5.442$, the area above the x-axis from $x = 4$ to $x = 5.442$ equals $\frac{20}{3}$, offsetting the area below from $x = 0$ to $x = 4$.

$b = 5.442$ is outside $(0, 4)$; it is the value where the positive area between $x = -2$ and $x = b$ above the x-axis equals the area below the x-axis.

Question 6

Worked Solution

$y = x(x+4)(x-2) = x^3 + 2x^2 - 8x$; crosses x-axis at $x = 0$, $x = -4$, $x = 2$.

(a)

A has $x = -4$, B has $x = 2$

(b) Two separate regions: $[-4, 0]$ (above x-axis) and $[0, 2]$ (below x-axis).

$\int (x^3 + 2x^2 - 8x)\,\mathrm{d}x = \frac{x^4}{4} + \frac{2x^3}{3} - 4x^2$

Region $[-4, 0]$:

$$\left[\frac{x^4}{4} + \frac{2x^3}{3} - 4x^2\right]_{-4}^{0} = 0 - \left(64 - \frac{128}{3} - 64\right) = 0 - \left(-\frac{128}{3}\right) = \frac{128}{3}$$

Region $[0, 2]$ (take absolute value):

$$\left|\left[\frac{x^4}{4} + \frac{2x^3}{3} - 4x^2\right]_{0}^{2}\right| = \left|4 + \frac{16}{3} - 16\right| = \left|-12 + \frac{16}{3}\right| = \left|-\frac{20}{3}\right| = \frac{20}{3}$$

Total area $= \frac{128}{3} + \frac{20}{3} = \frac{148}{3}$

Total area $= \frac{148}{3}$

Question 7

Worked Solution

$y = 10 + 8x + x^2 - x^3$. Maximum turning point at A.

(a) Show x-coordinate of A is 2:

$$\frac{dy}{dx} = 8 + 2x - 3x^2$$

At $x = 2$: $8 + 4 - 12 = 0$ ✓. Second derivative: $2 - 6x$; at $x = 2$: $2 - 12 = -10 < 0$ (maximum). ✓

$x = 2$ is the x-coordinate of A (shown)

(b) At $x = 2$: $y = 10 + 16 + 4 - 8 = 22$. So $A = (2, 22)$.

Region R is bounded by the curve, the y-axis and the line OA.

Line OA: gradient $= \dfrac{22 - 0}{2 - 0} = 11$, so $y = 11x$.

Area $= \displaystyle\int_0^2 [(10 + 8x + x^2 - x^3) - 11x]\,dx = \int_0^2 (10 - 3x + x^2 - x^3)\,dx$

$$= \left[10x - \frac{3x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}\right]_0^2 = 20 - 6 + \frac{8}{3} - 4 = 10 + \frac{8}{3} = \frac{38}{3}$$

Area of $R = \dfrac{38}{3}$

Question 8

Worked Solution

Curve C: $y = \frac{1}{8}x^3 + \frac{3}{4}x^2$. Max at A ($x = -4$), min at O. Line l touches C at A and cuts C at B ($x = 2$). Find area of region R.

Find line l: gradient at A (where $x = -4$, $y = \frac{-64}{8} + \frac{3 \cdot 16}{4} = -8 + 12 = 4$):

$$\frac{dy}{dx} = \frac{3x^2}{8} + \frac{3x}{2}$$

At $x = -4$: $\frac{48}{8} - 6 = 6 - 6 = 0$.

So l is horizontal: $y = 4$.

Area:

$$\text{Area} = \int_{-4}^{2} \left[4 - \left(\frac{x^3}{8} + \frac{3x^2}{4}\right)\right] dx = \int_{-4}^{2} \left(4 - \frac{x^3}{8} - \frac{3x^2}{4}\right) dx$$

$$= \left[4x - \frac{x^4}{32} - \frac{x^3}{4}\right]_{-4}^{2}$$

Upper ($x = 2$): $8 - \frac{16}{32} - 2 = 8 - \frac{1}{2} - 2 = \frac{11}{2}$

Lower ($x = -4$): $-16 - \frac{256}{32} + \frac{64}{4} = -16 - 8 + 16 = -8$

Area $= \frac{11}{2} - (-8) = \frac{11}{2} + 8 = \frac{27}{2}$

Area of $R = \frac{27}{2}$

Question 9

Worked Solution

Line: $y = 10 - x$. Curve: $y = 10x - x^2 - 8$.

(a) Find intersection points A and B:

$$10 - x = 10x - x^2 - 8 \implies x^2 - 11x + 18 = 0 \implies (x-2)(x-9) = 0$$

$x = 2 \Rightarrow y = 8 \quad$ and $\quad x = 9 \Rightarrow y = 1$.

$A = (2, 8)$, $B = (9, 1)$

(b) Area of R (curve above line):

The curve $y = 10x - x^2 - 8$ is above $y = 10 - x$ between $x = 2$ and $x = 9$.

$$\int_2^9 [(10x - x^2 - 8) - (10 - x)]\,dx = \int_2^9 (11x - x^2 - 18)\,dx$$

$$= \left[\frac{11x^2}{2} - \frac{x^3}{3} - 18x\right]_2^9$$

Upper $(x = 9)$: $\frac{891}{2} - 243 - 162 = \frac{891}{2} - 405 = \frac{891 - 810}{2} = \frac{81}{2}$

Lower $(x = 2)$: $22 - \frac{8}{3} - 36 = -14 - \frac{8}{3} = -\frac{50}{3}$

Area $= \frac{81}{2} - \left(-\frac{50}{3}\right) = \frac{81}{2} + \frac{50}{3} = \frac{243}{6} + \frac{100}{6} = \frac{343}{6}$

Area of $R = \frac{343}{6}$

Question 10

Worked Solution

Line: $y = x + 4$. Curve: $y = -x^2 + 2x + 24$.

(a) Find A and B:

$$-x^2 + 2x + 24 = x + 4 \implies x^2 - x - 20 = 0 \implies (x-5)(x+4) = 0$$

$x = 5 \Rightarrow y = 9$; $x = -4 \Rightarrow y = 0$.

$A = (-4, 0)$, $B = (5, 9)$

(b) Area of R:

$$\int_{-4}^{5} [(-x^2 + 2x + 24) - (x + 4)]\, dx = \int_{-4}^{5} (-x^2 + x + 20)\, dx$$

$$= \left[-\frac{x^3}{3} + \frac{x^2}{2} + 20x \right]_{-4}^{5}$$

Upper $(x = 5)$: $-\frac{125}{3} + \frac{25}{2} + 100 = \frac{-250 + 75 + 600}{6} = \frac{425}{6}$

Lower $(x = -4)$: $\frac{64}{3} + 8 - 80 = \frac{64}{3} - 72 = \frac{64 - 216}{3} = -\frac{152}{3}$

Area $= \frac{425}{6} - \left(-\frac{152}{3}\right) = \frac{425}{6} + \frac{304}{6} = \frac{729}{6} = \frac{243}{2}$

Area of $R = \frac{243}{2} = 121.5$

Question 11

Worked Solution

Curve C: $y = 6x - x^2$. Line L: $y = 2x$.

(a) C intersects x-axis where $y = 0$: $x(6 - x) = 0 \Rightarrow x = 0$ or $x = 6$. ✓

(b) Intersection of C and L: $6x - x^2 = 2x \Rightarrow x^2 - 4x = 0 \Rightarrow x(x - 4) = 0$, so $x = 0$ or $x = 4$. When $x = 4$: $y = 8$. Points $(0, 0)$ and $(4, 8)$. ✓

(c) Area of R (between C and L, from $x = 0$ to $x = 4$):

$$\int_0^4 [(6x - x^2) - 2x]\,\mathrm{d}x = \int_0^4 (4x - x^2)\,\mathrm{d}x = \left[2x^2 - \frac{x^3}{3}\right]_0^4 = 32 - \frac{64}{3} = \frac{96 - 64}{3} = \frac{32}{3}$$

Area of $R = \dfrac{32}{3}$

Question 12

Worked Solution

Curve: $y = 4x^3 + 9x^2 - 30x - 8$. Turning point A at $x = 1$, $B = (2, 0)$, $C = \left(-\frac{1}{4}, 0\right)$.

(a) Show x-coordinate of A is 1:

$$\frac{dy}{dx} = 12x^2 + 18x - 30$$

At $x = 1$: $12 + 18 - 30 = 0$ ✓

$x = 1$ is a turning point (shown)

(b) $A = (1, 4 + 9 - 30 - 8) = (1, -25)$.

Line AB passes through $A(1, -25)$ and $B(2, 0)$: gradient $= \dfrac{0 - (-25)}{2 - 1} = 25$.

$$y - 0 = 25(x - 2) \implies y = 25x - 50$$

Region R is bounded by the curve, line AB, and the x-axis between $x = -\frac{1}{4}$ and $x = 2$.

Split into two parts. The curve lies below the x-axis from $x = -\frac{1}{4}$ to $x = 2$ (since A is a minimum at $y = -25$).

Area of triangle ABP where $P = (1, 0)$: $\frac{1}{2} \times 1 \times 25 = 12.5$

Area between curve and x-axis (from $x = -\frac{1}{4}$ to $x = 2$):

$$\left| \int_{-\frac{1}{4}}^{2} (4x^3 + 9x^2 - 30x - 8)\, dx \right|$$

$$= \left| \left[x^4 + 3x^3 - 15x^2 - 8x\right]_{-\frac{1}{4}}^{2} \right|$$

Upper ($x = 2$): $16 + 24 - 60 - 16 = -36$

Lower ($x = -\frac{1}{4}$): $\dfrac{1}{256} - \dfrac{3}{64} - \dfrac{15}{16} + 2 = \dfrac{1}{256} - \dfrac{12}{256} - \dfrac{240}{256} + \dfrac{512}{256} = \dfrac{261}{256}$

$\left| -36 - \dfrac{261}{256} \right| = 36 + \dfrac{261}{256} \approx 36 + 1.02 = 37.02$

Area of region $R = 37.02 - 12.5 = 24.52 \approx \mathbf{24.52}$

Exact value: $36 + \dfrac{261}{256} - \dfrac{25}{2} = \dfrac{9216 + 261 - 3200}{256} - \dfrac{5125}{256}$

Let me recompute carefully.

$F(2) = 16 + 24 - 60 - 16 = -36$

$F(-\frac{1}{4}) = \frac{1}{256} + 3(-\frac{1}{64}) - 15(\frac{1}{16}) - 8(-\frac{1}{4}) = \frac{1}{256} - \frac{12}{256} - \frac{240}{256} + \frac{512}{256} = \frac{261}{256}$

Area under x-axis $= |F(2) - F(-\frac{1}{4})| = |-36 - \frac{261}{256}| = 36 + \frac{261}{256} = \frac{9216+261}{256} = \frac{9477}{256}$

Area of triangle $= \frac{1}{2}(1)(25) = \frac{25}{2}$

Area of $R = \frac{9477}{256} - \frac{25}{2} = \frac{9477}{256} - \frac{3200}{256} = \frac{6277}{256} \approx 24.52$

Area of $R \approx 24.52$ (to 2 d.p.)

End of Worked Solutions