Simple, Forced and Damped Harmonic Motion Exam Questions (From OCR 4758)

Q1, (Jan 2006, Q1)

In an electric circuit, the current, I amps, at time t seconds is modelled by the differential equation

$$\frac{d^2I}{dt^2} + 6\frac{dI}{dt} + kI = 6e^{-t},$$

where k is a positive constant which depends on the capacitor in the circuit.

(i) In the case $k = 8$, find the general solution. [8]

(ii) In the case $k = 9$, find the solution given that initially the current is 1.5 amps and $\frac{dI}{dt} = 0$. State the limiting value of the current as t tends to infinity. [12]

(iii) Show that, for all positive values of k, the complementary function for this differential equation will tend to zero as t tends to infinity. [4]

Q2, (Jun 2006, Q1)

The displacement x at time t of an oscillating system from a fixed point is given by

$$\ddot{x} + 2\lambda\dot{x} + 5x = 0,$$

where $\lambda \geqslant 0$.

(i) For what value of λ is the motion simple harmonic? State the general solution in this case. [3]

(ii) Find the range of values of λ for which the system is under-damped. [3]

Consider the case $\lambda = 1$.

(iii) Find the general solution of the differential equation. [3]

When $t = 0$, $x = x_0$ and $\dot{x} = 0$, where x_0 is a positive constant.

(iv) Find the particular solution. [4]

(v) Find the least positive value of t for which $x = 0$. [3]

Now consider the case $\lambda = 3$ with the same initial conditions.

(vi) Find the particular solution and show that it is never zero for $t > 0$. [8]

Q3, (Jun 2007, Q1)

An object is suspended from one end of a vertical spring in a resistive medium. The other end of the spring is made to oscillate and the differential equation describing the motion of the object is

$$\ddot{y} + 4\dot{y} + 29y = 3\cos t,$$

where y is the displacement at time t of the object from its equilibrium position.

(i) Find the general solution. [11]

(ii) Find the particular solution subject to the conditions $\dot{y} = y = 0$ when $t = 0$. What is the amplitude of the motion for large values of t? [8]

(iii) Find the displacement and velocity of the object when $t = 10\pi$. [2]

At $t = 10\pi$, the upper end of the spring stops oscillating and the differential equation describing the motion of the object is now

$$\ddot{y} + 4\dot{y} + 29y = 0.$$

(iv) Write down the general solution. Describe briefly the motion for $t > 10\pi$. [3]

Q4, (Jun 2008, Q1)

Fig. 1 shows a particle of mass 2 kg suspended from a light vertical spring. At time t seconds its displacement is x m below its equilibrium level and its velocity is v m s^{-1} vertically downwards. The forces on the particle are

- its weight, $2g$ N
- the tension in the spring, $8(x + 0.25g)$ N
- the resistance to motion, $2kv$ N where k is a positive constant.

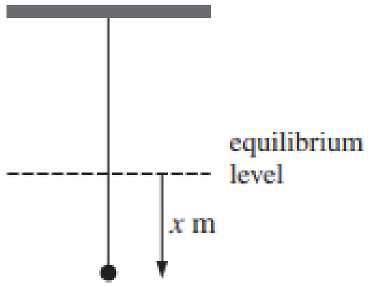

Fig. 1

(i) Use Newton's second law to write down the equation of motion for the particle, justifying the signs of the terms. Hence show that the displacement is described by the differential equation

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + 4x = 0. \quad [4]$$

The particle is initially at rest with $x = 0.1$.

(ii) In the case $k = 0$, state the general solution of the differential equation. Find the solution, subject to the given initial conditions. [4]

(iii) In the case $k = 2$, find the solution of the differential equation, subject to the given initial conditions. Sketch a graph of the solution for $t \geqslant 0$. [11]

(iv) Find the range of values of k for which the system is over-damped. Sketch a possible graph of the solution in such a case. [5]

Q5, (Jun 2009, Q1)

A car travels over a rough surface. The vertical motion of the front suspension is modelled by the differential equation

$$\frac{d^2y}{dt^2} + 25y = 20\cos 5t,$$

where y is the vertical displacement of the top of the suspension and t is time.

(i) Find the general solution. **[8]**

Initially $y = 1$ and $\frac{dy}{dt} = 0$.

(ii) Find the solution subject to these conditions. **[4]**

(iii) Sketch the solution curve for $t \geqslant 0$. **[4]**

A refined model of the motion of the suspension is given by

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 25y = 20\cos 5t.$$

(iv) Verify that $y = 2\sin 5t$ is a particular integral for this differential equation. Hence find the general solution. **[6]**

(v) Compare the behaviour of the suspension predicted by the two models. **[2]**

Q6, (Jan 2010, Q1)

A particle is attached to a spring and suspended vertically from an oscillating platform. The vertical displacement, y, of the particle from a fixed point at time t is modelled by the differential equation

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 0.5\sin t.$$

(i) Find the general solution. **[9]**

Initially the displacement and velocity are both zero.

(ii) Find the solution. **[5]**

(iii) Describe the motion of the particle for large values of t. **[2]**

(iv) Find approximate values of the velocity and displacement at $t = 20\pi$. **[3]**

The motion of the platform is stopped at $t = 20\pi$ and the differential equation modelling the subsequent motion of the particle is

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 0.$$

(v) Write down the general solution. Sketch the solution curve for $t > 20\pi$. **[5]**

Q7, (Jun 2013, Q1)

A particle is attached to a spring and suspended vertically from a point P which is made to oscillate vertically. The vertical displacement, x, of the particle from a fixed point at time t is modelled by the differential equation

$$2\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + x = \cos t.$$

(i) Find the general solution of the differential equation. **[8]**

Initially the displacement and velocity of the particle are both zero.

(ii) Find the particular solution and sketch its graph for large positive values of t. **[6]**

(iii) Find approximate values of the displacement and velocity at $t = 10\pi$. **[3]**

The point P stops oscillating at $t = 10\pi$ and the subsequent motion of the particle is modelled by

$$2\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + x = 0.$$

(iv) Determine the type of damping present. **[2]**

(v) Using the values obtained in part **(iii)**, find the particular solution for this motion. **[5]**

Q8, (Jun 2015, Q1)

The displacement, x m, of a particle at time t s is given by the differential equation

$$\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 25x = 0.$$

Initially the particle is at the origin and has a velocity of $\frac{1}{4}$ms^{-1}.

(i) Find the particular solution for x. **[8]**

(ii) Find the maximum displacement of the particle from its initial position, giving your answer correct to 3 significant figures. **[4]**

(iii) Describe the behaviour of your solution for large values of t. **[1]**

In a different situation, an additional force is applied to the particle and the differential equation satisfied by x is

$$\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 25x = 5\sin 5t.$$

(iv) Using the same initial conditions as in part **(i)**, find the new particular solution for x. **[10]**

(v) Describe the behaviour of your new solution for large values of t. **[1]**
