Proof By Induction (Challenging) Exam Questions

Q1, (Cambridge 9795/01, 2011 Specimen, Q13)

Let $I_n = \int_1^e (\ln x)^n \, dx$, where n is a positive integer.

(i) By considering $\frac{d}{dx}\left(x(\ln x)^n\right)$, or otherwise, show that $I_n = e - nI_{n-1}$. [4]

(ii) Let $J_n = \frac{I_n}{n!}$. Prove by induction that

$$\sum_{r=2}^{n} \frac{(-1)^r}{r!} = \frac{1}{e}\left(1 + (-1)^n J_n\right)$$

for all positive integers $n \geqslant 2$. [10]

Q2, (Cambridge 9795/01, 2013, Q12)

Given $y = xe^{2x}$,

(i) find the first four derivatives of y with respect to x, [4]

(ii) conjecture an expression for $\frac{d^n y}{dx^n}$ in the form $(ax + b)e^{2x}$, where a and b are functions of n, [2]

(iii) prove by induction that your result holds for all positive integers n. [5]

Q3, (Cambridge 9795/01, 2016, Q11)

(i) The sequence of *Fibonacci Numbers* $\{F_n\}$ is given by

$$F_1 = 1, \quad F_2 = 1 \quad \text{and} \quad F_{n+1} = F_n + F_{n-1} \quad \text{for } n \geqslant 2.$$

Write down the values of F_3 to F_6. [1]

(ii) The sequence of functions $\{p_n(x)\}$ is given by

$$p_1(x) = x + 1 \quad \text{and} \quad p_{n+1}(x) = 1 + \frac{1}{p_n(x)} \quad \text{for } n \geqslant 1.$$

(a) Find $p_2(x)$ and $p_3(x)$, giving each answer as a single algebraic fraction, and show that $p_4(x) = \frac{3x+5}{2x+3}$. [3]

(b) Conjecture an expression for $p_n(x)$ as a single algebraic fraction involving Fibonacci numbers, and prove it by induction for all integers $n \geqslant 2$. [5]

Q4, (Cambridge 9795/01, 2018, Q8)

(i) Write down the values of the constants a and b for which $m^5 \equiv \frac{1}{6}m^3(am^2 + 2) - \frac{1}{12}m^2(bm)$. [1]

(ii) Prove by induction that $\sum_{r=1}^{n} r^5 = \frac{1}{6}n^3(n+1)^3 - \frac{1}{12}n^2(n+1)^2$ for all positive integers n. [7]

Q5, (Cambridge 9795/01, 2017, Q12)

For each positive integer n, the function F_n is defined for all real angles θ by

$$\mathrm{F}_n(\theta) = c^{2n} + s^{2n}$$

where $c = \cos\theta$ and $s = \sin\theta$.

(i) Prove the identity

$$\mathrm{F}_{n+2}(\theta) - \tfrac{1}{4}\sin^2 2\theta \times \mathrm{F}_{n+1}(\theta) \equiv \mathrm{F}_{n+3}(\theta).$$ [4]

Let z denote the complex number $c + \mathrm{i}s$.

(ii) Using de Moivre's theorem,

(a) express $z + z^{-1}$ and $z - z^{-1}$ in terms of c and s respectively, [3]

(b) prove the identity $8(c^6 + s^6) \equiv 3\cos 4\theta + 5$ and deduce that

$$c^6 + s^6 \equiv \cos^2 2\theta + \tfrac{1}{4}\sin^2 2\theta.$$ [7]

(iii) Prove by induction that, for all positive integers n,

$$c^{2n+4} + s^{2n+4} \leqslant \cos^2 2\theta + \frac{1}{2^{n+1}}\sin^2 2\theta.$$

[You are **given** that the range of the function F_n is $\frac{1}{2^{n-1}} \leqslant \mathrm{F}_n(\theta) \leqslant 1$.] [7]
