DeMoivre's Theorem and Applications To Trigonometry (From OCR 4727)

Q1, (Jun 2007, Q5)

(i) Use de Moivre's theorem to prove that

$$\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1. \qquad [4]$$

(ii) Hence find the largest positive root of the equation

$$64x^6 - 96x^4 + 36x^2 - 3 = 0,$$

giving your answer in trigonometrical form. [4]

Q2, (Jun 2008, Q4)

(i) By expressing $\cos\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$, show that

$$\cos^5\theta \equiv \tfrac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta). \qquad [5]$$

(ii) Hence solve the equation $\cos 5\theta + 5\cos 3\theta + 9\cos\theta = 0$ for $0 \leqslant \theta \leqslant \pi$. [4]

Q3, (Jan 2009, Q8)

(i) By expressing $\sin\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$, show that

$$\sin^6\theta \equiv -\tfrac{1}{32}(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10). \qquad [5]$$

(ii) Replace θ by $\left(\tfrac{1}{2}\pi - \theta\right)$ in the identity in part **(i)** to obtain a similar identity for $\cos^6\theta$. **[3]**

(iii) Hence find the exact value of $\displaystyle\int_0^{\frac{1}{4}\pi} \left(\sin^6\theta - \cos^6\theta\right) d\theta$. **[4]**

Q4, (Jan 2010, Q7)

(i) Solve the equation $\cos 6\theta = 0$, for $0 < \theta < \pi$. **[3]**

(ii) By using de Moivre's theorem, show that

$$\cos 6\theta \equiv (2\cos^2\theta - 1)(16\cos^4\theta - 16\cos^2\theta + 1). \qquad \textbf{[5]}$$

(iii) Hence find the exact value of

$$\cos\left(\tfrac{1}{12}\pi\right)\cos\left(\tfrac{5}{12}\pi\right)\cos\left(\tfrac{7}{12}\pi\right)\cos\left(\tfrac{11}{12}\pi\right),$$

justifying your answer. **[5]**

Q5, (Jun 2013, Q8)

(i) Use de Moivre's theorem to show that $\cos 5\theta \equiv 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$. **[5]**

(ii) Hence find the roots of $16x^4 - 20x^2 + 5 = 0$ in the form $\cos\alpha$ where $0 \leqslant \alpha \leqslant \pi$. **[4]**

(iii) Hence find the exact value of $\cos\frac{1}{10}\pi$. **[3]**

Q6, (Jan 2013, Q7)

Let $S = e^{i\theta} + e^{2i\theta} + e^{3i\theta} + \ldots + e^{10i\theta}$.

(i) (a) Show that, for $\theta \neq 2n\pi$, where n is an integer,

$$S = \frac{e^{\frac{1}{2}i\theta}\left(e^{10i\theta} - 1\right)}{2i\sin\left(\frac{1}{2}\theta\right)}.$$ **[4]**

(b) State the value of S for $\theta = 2n\pi$, where n is an integer. **[1]**

(ii) Hence show that, for $\theta \neq 2n\pi$, where n is an integer,

$$\cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos 10\theta = \frac{\sin\left(\frac{21}{2}\theta\right)}{2\sin\left(\frac{1}{2}\theta\right)} - \frac{1}{2}.$$ **[3]**

(iii) Hence show that $\theta = \frac{1}{11}\pi$ is a root of $\cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos 10\theta = 0$ and find another root in the interval $0 < \theta < \frac{1}{4}\pi$. **[4]**

Q7, (Jun 2014, Q7)

(i) By expressing $\sin\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$, show that

$$\sin^5\theta \equiv \tfrac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta).$$ **[4]**

(ii) Hence solve the equation

$$\sin 5\theta + 4\sin\theta = 5\sin 3\theta$$

for $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$. **[4]**

Q8, (Jun 2016, Q7)

(i) Use de Moivre's theorem to show that

$$\sin 6\theta \equiv \cos\theta(6\sin\theta - 32\sin^3\theta + 32\sin^5\theta).$$ **[5]**

(ii) Hence show that, for $\sin 2\theta \neq 0$,

$$-1 \leqslant \frac{\sin 6\theta}{\sin 2\theta} < 3.$$ **[7]**

Q9, (Jun 2017, Q7)

(i) By expressing $\cos\theta$ in terms of $e^{i\theta}$ show that

$$\cos^6\theta = \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10).$$ **[4]**

(ii) Hence solve, for $0 \leqslant \theta \leqslant \pi$,

$$\cos 6\theta + 6\cos 4\theta + 2\cos 2\theta = 3.$$ **[5]**
